1 The Euclidean Plane

Definition. The *Euclidean Distance* $d(P_1, P_2)$ between points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ is $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The Euclidean distance satisfies the three properties:

(a) For any pair of points $P, Q \in \mathbb{R}^2, d(P, Q) \geq 0$, and equality only occurs if $P = Q$.

(b) For any pairs of points $P, Q \in \mathbb{R}^2, d(P, Q) = d(Q, P)$.

(c) For any three points $P, Q, R \in \mathbb{R}^2, d(P, Q) \leq d(Q, R) + d(R, P)$, and equality only occurs if R lies on the segment $\overline{PQ}$.

Definition. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an *isometry* of $\mathbb{R}^2$ if $d(f(P_1), f(P_2)) = d(P_1, P_2)$ for all $P_1, P_2 \in \mathbb{R}^2$.

***Definition.** The three basic isometries on $\mathbb{R}^2$ are the translation by (α, β), $t_{(\alpha,\beta)}(x, y) = (x+\alpha, y+\beta)$, the reflection in the x-axis, $\overline{r}(x, y) = (x, -y)$, and the rotation about the origin at angle $\theta \in S^1$ (note $S^1 = [0, 2\pi]/\{0 \sim 2\pi\}$), $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Some basic properties of these three are:

- $R_{\theta,P} \circ R_{\phi,P} = R_{\theta+\phi,P}$
- $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)}$
- $\overline{r}_L \circ \overline{r}_L = \text{Id}$.

Definition. The rotation about the point (α, β) by angle θ is formed by conjugation: $R_{\theta,(\alpha,\beta)} = t_{(\alpha,\beta)} R_\theta t_{(-\alpha,-\beta)}$, and the reflection across the line L not going through the origin is $\overline{r}_L = tR \circ \overline{r} \circ R^{-1}t^{-1}$, where t^{-1} translates L to go through $(0, 0)$, R^{-1} rotates L to the x-axis, and tR rotates and translates back to the initial position.

***Theorem.** Any translation or rotation is the product of two reflections. Conversely, the product of two reflections is a rotation or translation.
(The Theorem could also be read as: An isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a translation or rotation if and only if it is the product of two reflections.)
In particular, a translation by δ units is the product of two reflections across parallel lines (perpendicular to direction of translation) that are distance $\frac{\delta}{2}$ apart ($t_{(0,\delta)} = t_{(0,\delta/2)}\overline{r}t_{(0,\delta/2)^{-1}}\overline{r}$), and a rotation by θ about a point P is the product of two reflections across lines intersecting at point P with angle of intersection $\frac{\theta}{2}$ ($R_\theta = R_{\theta/2}\overline{r}R_{\theta/2}^{-1}\overline{r}$).

Corollary. If $\overline{r}_M\overline{r}_L$ is a rotation, then $\overline{r}_{M'}\overline{r}_{L'} = \overline{r}_M\overline{r}_L$ for any lines L', M' with the same intersection as L, M and the same signed angle from L to M. If $\overline{r}_M\overline{r}_L$ is a translation, then $\overline{r}_{M'}\overline{r}_{L'} = \overline{r}_M\overline{r}_L$ for any lines L', M' parallel to L, M and the same signed distance apart.

***Corollary.** The set of translations and rotations is closed under product. In particular, translations composed with translations is another translation; the composition of two rotations about the same point is a rotation about the same point; the composition of two rotations at different points with angle θ, ϕ is a rotation if $\theta/2 + \phi/2 \not\equiv \pi \pmod{2\pi}$, and is a translation if $\theta/2 + \phi/2 \equiv \pi \pmod{2\pi}$.

***Lemma.** Any isometry f of $\mathbb{R}^2$ is determined by the images $f(A), f(B), f(C)$ of three **noncollinear** points $A, B, C \in \mathbb{R}^2$.

Corollary. If L is the line of points equidistant from P, Q, then $\overline{r}_L(P) = Q, \overline{r}_L(Q) = P$.

***Theorem.** Any isometry f of $\mathbb{R}^2$ is the product of one, two, or three reflections.

***Definition.** A set G with an operation $\circ$ form a group $(G, \circ)$ if it satisfies:

- **Closure:** If $a, b \in G$, then $a \circ b \in G$.
- **Associativity:** For all $a, b, c \in G, (a \circ b) \circ c = a \circ (b \circ c)$. You may assume function composition is associative.
- **Identity:** There exists an identity element Id$\in G$ such that for all $a \in G, a \circ \mathrm{Id} = a = \mathrm{Id} \circ a$.
- **Inverses:** For all $a \in G$, there exists $b \in G$ such that $a \circ b = \mathrm{Id} = b \circ a$ (usually will denote $b = a^{-1}$).

Corollary. The isometries of $\mathbb{R}^2$ form a group Iso($\mathbb{R}^2$), and the products of even numbers of reflections form a subgroup Iso$^+(\mathbb{R}^2)$. The products of odd numbers of reflections do not form a group (closure fails).

***Definition.** *Orientation-preserving isometries* are even products of reflections, i.e. translations and rotations. *Orientation-reversing isometries* are odd products of reflections, i.e. reflections and glide reflections. A *glide reflection* is an isometry of the form $t_{(\alpha,\beta)}\overline{r}_L$, where $t_{(\alpha,\beta)}$ is a translation in the direction of L. The translation and reflection of a glide reflection commute, i.e. $t_{(\alpha,\beta)} \circ \overline{r}_L = \overline{r}_L \circ t_{(\alpha,\beta)}$

Theorem. A product $\overline{r}_N\overline{r}_M\overline{r}_L$ of reflections in lines L, M, N is a glide reflection.

***Theorem. Classification of Euclidean Isometries.** Each isometry of $\mathbb{R}^2$ is either a rotation, translation, or glide reflection.

2 Euclidean Surfaces

Definition. The group generated by the isometry f is $\Gamma = \langle f \rangle = \{f^n : n \in \mathbb{Z}\}$. The group generated by two isometries f, g is $\Gamma' = \langle f, g \rangle = \left\{ \prod_{i=1}^{k} f^{n_i} g^{m_i} : n_i, m_i \in \mathbb{Z} \right\}$. That is, any element $h \in \Gamma'$ is of the form $h = f^{n_k} g^{m_k} f^{n_{k-1}} g^{m_{k-1}} \cdots f^{n_2} g^{m_2} f^{n_1} g^{m_1}$.

Definition. For a group Γ, the *quotient* $\mathbb{R}^2/\Gamma$ is the space consisting of equivalence classes of points in $\mathbb{R}^2$, where we identify $(x, y) \sim f(x, y)$ for any $f \in \Gamma$.

***Definition.** The *cylinder* is $C = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \rangle = \{t_{(n,0)} : n \in \mathbb{Z}\}$. In particular, for $(x, y) \sim (x + n, y)$ for any $n \in \mathbb{Z}$. A *point* of C is a set of the form $\{(x + n, y) : n \in \mathbb{Z}\}$, called the Γ-*orbit* of (x, y). That is, if $P = (x, y) \in \mathbb{R}^2$, then its Γ-orbit consists of all the points equivalent to P in C. So a single point in a quotient space is a set containing infinitely many points that are identified with each other via Γ. Typically, we visualize a quotient by looking at a *fundamental domain*, which is a part of $\mathbb{R}^2$ which contains a representative of each Γ-orbit in its interior (the boundary typically has some sort of identification, e.g. for C, the strip $0 \leq x \leq 1$ is a fundamental domain, and the boundary $x = 0, x = 1$ are equivalent lines in C.) A *line* in C is the image of a line $L \subseteq \mathbb{R}^2$ under the projection map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$.

***Definition.** The *cylindrical distance* between two points $\Gamma(P), \Gamma(Q)$ is the minimum Euclidean distance between P', Q' among all representatives in the equivalence class $\Gamma(P), \Gamma(Q)$. i.e. $d_C(\Gamma(P), \Gamma(Q)) = \min\{d(P', Q') : P' \in \Gamma(P), Q' \in \Gamma(Q)\} = \min\{d(P, Q') : Q \in \Gamma(Q)\}$. The same formula is used in other quotient spaces $\mathbb{R}^2/\Gamma$ as well.

Definition. The *Möbius Band* is $M = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \circ \overline{r} \rangle$. In particular, $(x, y) \sim (x + n, (-1)^n y)$ for any $n \in \mathbb{Z}$.

Definition. The *Torus* is $T^2 = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(0,1)}, t_{(0,1)} \rangle = \{t_{(n,m)} : n, m \in \mathbb{Z}\}$. In particular, $(x, y) \sim (x + n, y + m)$ for any $(n, m) \in \mathbb{Z}$.

Definition. The *Klein Bottle* is $K = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \circ \overline{r} \rangle$. In particular, $(x, y) \sim (x + n, (-1)^n y)$ and $(x, y) \sim (x, y + n)$ for any $n \in \mathbb{Z}$.

Definition. Γ is *discontinuous* if no $P \in \mathbb{R}^2$ has a Γ-orbit with a limit point (i.e. a point whose neighborhoods all include infinitely many points of $\Gamma(P)$). Γ is *fixed point free* if every nontrivial element (the trivial element is Id) in Γ has no fixed points.

Lemma. If Γ is a group of isometries of $\mathbb{R}^2$, then Γ is discontinuous and fixed point free if and only if each $P \in \mathbb{R}^2$ has a neighborhood D_P in which each point belongs to a different Γ-orbit.

Theorem. A discontinuous, fixed point free group Γ of isometries of $\mathbb{R}^2$ is generated by one or two elements.

Corollary. If Γ is generated by a single translation, then $\mathbb{R}^2/\Gamma$ is a cylinder. If Γ is generated by a single glide reflection, then $\mathbb{R}^2/\Gamma$ is a Möbius band. If Γ is generated by two translations (in different directions), then $\mathbb{R}^2/\Gamma$ is a Torus. If Γ is generated by one translation and one glide reflection, or by two glide reflections, then $\mathbb{R}^2/\Gamma$ is a Klein bottle.

3 The Sphere

***Definition.** The *Sphere* is $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. The *Spherical Distance* between $P_1, P_2 \in S^2$ is given by $d_{S^2}(P_1, P_2) = \theta = 2 \sin^{-1} \frac{1}{2} d(P_1, P_2)$, where θ is the angle between the two vectors $\langle P_1 \rangle, \langle P_2 \rangle$.

Definition. The *rotation about the z-axis* is $R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Similarly, we have the *rotations about the x-axis and y-axis* as $R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$. The *reflection in the xy-plane (i.e. z=0)* is $\overline{r}_E(x, y, z) = (x, y, -z)$.

Lemma. The set of points equidistant from two points $P, P' \in S^2$ is a plane through $(0, 0, 0)$, and a reflection in this plane exchanges P, P'.

***Definition.** A *line* in S^2 is the set of points equidistant to two points $P, P' \in S^2$. Equivalently, a line is the intersection of S^2 with a plane through $(0, 0, 0)$. We call these *great circles.*

***Theorem.** Any isometry of S^2 is the product of one, two, or three reflections.

Corollary. The isometries of S^2 form a group, $\text{Iso}(S^2)$.

Lemma. A rotation $R_{K,\theta}$ in $\mathbb{R}^3$ about any line K (here a line is in $\mathbb{R}^3$, e.g. the z-axis, rather than a great circle) through angle θ is the product of reflections in any planes Π, Π' through K separated by angle $\theta/2$. (This is synonymous to rotations in $\mathbb{R}^2$ being products of two reflections across lines intersecting at angle $\theta/2$). This lemma may be re-expressed in spherical terms: A rotation $R_{P,\theta}$ of S^2 about P through angle θ is the product of reflections in any lines (great circles) $L, L' \subseteq S^2$ through P separated by angle $\theta/2$.

Theorem. The product of two rotations of S^2 is also a rotation. The rotations of S^2 form a group. (Note that any rotation of S^2 fixes the origin, and that's why closure is satisfied). Furthermore, the rotations form the group $\text{Iso}^+(S^2)$ of orientation-preserving isometries.

***Lemma.** There is a unique line through any pair of points $P, P' \in S^2$ which are not antipodal points (two points P, P' are *antipodal* if $P = -P'$). (This lemma follows from the fact that if P, P' are not antipodal, then $P, P', (0, 0, 0) \in \mathbb{R}^3$ are noncollinear, hence from MAT 21C, we know there exists a unique plane through the three points, and its intersection with S^2 is the unique great circle through P, P'.)

4 The Hyperbolic Plane

***Definition.** The *Upper Half Plane* is $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\} = \{x + yi \in \mathbb{C} : y > 0\}$, with distance function defined by $ds^2 = \dfrac{dx^2 + dy^2}{y^2} \iff ds = \dfrac{\sqrt{dx^2 + dy^2}}{y} = \dfrac{\sqrt{1 + (dy/dx)^2}}{y}dx = \dfrac{\sqrt{(dx/dy)^2 + 1}}{y}dy = \dfrac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{y(t)}dt = \dfrac{|dz|}{\text{Im } z}$, and the length of a curve s is computed via $\int ds$. One may compute the length of a curve by writing the equation of the curve as $y = f(x), x = g(y)$, or parametrizing with $(x(t), y(t))$.

Definition. The *distance* between two points in $\mathbb{H}^2$ is the minimum length of the collection of all curves between the two points. In particular, the *line* through two points with the same x-value is a vertical line, and the line through two points with different x-values is a semicircle centered on the x-axis going through the two points. These lines are also called *geodesics*, and the distance between two points is the length of the segment of the geodesic between the two points. A special case is when the two points have the same x-value: the distance between (x, y_1) and (x, y_2) with $y_2 > y_1$ is $\ln\left(\frac{y_2}{y_1}\right)$.

There is the following formula for the hyperbolic distance between $z, w \in \mathbb{H}^2$:

$$d_{\mathbb{H}^2}(z,w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\ \mathrm{Im}(z)\ \mathrm{Im}(w)}\right).$$

We have not proven this formula in class, and you should not use it to solve problems, as it is unjustified. We nevertheless provide it here in case it is useful to verify your answers.

Construction. Given two points $P, Q \in \mathbb{H}^2$ such that Re $P \neq$ Re Q (if Re $P =$ Re Q, then the geodesic is $x =$ Re P), we may construct the geodesic through P, Q as follows: 1) Construct the perpendicular bisector of $\overline{PQ}$, and find the intersection point with the x-axis. This is the center of the geodesic. 2) Compute the distance between the intersection point and one of P, Q. This is the radius of the geodesic. 3) Use the equation of a circle with given center and radius to find the equation.
The special case is when we consider $P, Q \in \mathbb{R} \subseteq \partial\mathbb{H}^2$. Note that any geodesic will intersect $\partial\mathbb{H}^2$ twice, so such P, Q will always exist (if one of P, Q is the point at ∞, then the geodesic is just a vertical line). The geodesic through $P, Q \in \mathbb{R} \subseteq \partial\mathbb{H}^2$ is $\left\{(x,y) \in \mathbb{H}^2 : \left(x - \frac{P+Q}{2}\right)^2 + y^2 = \left(\frac{P-Q}{2}\right)^2\right\}$.

Remark. For a general half circle with center $(\alpha, 0)$ and radius r, we have $(x-\alpha)^2 + y^2 = r^2 \implies y = \sqrt{r^2 - (x-\alpha)^2}$, $\frac{dy}{dx} = \frac{-(x-\alpha)}{\sqrt{r^2-(x-\alpha)^2}}$, and the distance formula simplifies:

$$\int \frac{\sqrt{1+(dy/dx)^2}}{y}dx = \int \frac{\sqrt{1+\frac{(-(x-\alpha))^2}{r^2-(x-\alpha)^2}}}{\sqrt{r^2-(x-\alpha)^2}}dx = \int \frac{\sqrt{\frac{r^2}{r^2-(x-\alpha)^2}}}{\sqrt{r^2-(x-\alpha)^2}}dx = \int \frac{r}{r^2-(x-\alpha)^2}dx.$$

Alternatively, we can parametrize by $(x(t), y(t)) = (\alpha + r\cos t, r\sin t)$, so $\frac{dx}{dt} = -r\sin t$, $\frac{dy}{dt} = r\cos t$, and thus the distance formula simplifies:

$$\int \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{y(t)}dt = \int \frac{\sqrt{r^2\sin^2 t + r^2\cos^2 t}}{r\sin t}dt = \int \frac{1}{\sin t}dt = \ln\left(\tan\left(\frac{t}{2}\right)\right) + C.$$

Note that in the parametrization, the center and radius do not matter for computing the integral (they matter when we compute the bounds of the integration).

The following properties from Euclidean Geometry also hold in Hyperbolic Geometry:

- *The Triangle Inequality*: If $P, Q, R \in \mathbb{H}^2$, then ($\mathbb{H}^2$-length of PR)+($\mathbb{H}^2$-length of RQ) $\geq$ ($\mathbb{H}^2$-length of PQ).
- *Set of Points Equidistant from Two points*: The set of points $\mathbb{H}^2$-equidistant from two points $P, P' \in \mathbb{H}^2$ is an $\mathbb{H}^2$-line L, and $\mathbb{H}^2$-reflection in L exchanges P, P'.
- *Three Reflections Theorem*: Each $\mathbb{H}^2$-isometry is the product of one, two, or three $\mathbb{H}^2$-reflections.
- The $\mathbb{H}^2$-isometries form a group, and the orientation-preserving and orientation-reversing $\mathbb{H}^2$-isometries form complementary sets.

***Theorem.** The orientation-preserving $\mathbb{H}^2$-isometries are of the form

$$f(z) = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$. The orientation-reversing $\mathbb{H}^2$-isometries are of the form

$$\overline{f}(z) = \frac{-a\overline{z}+b}{-c\overline{z}+d}$$

for $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
Some of the basic $\mathbb{H}^2$-isometries are:

- "Translation" (Limit Rotation) $t_\alpha(z) = z + \alpha$, $\alpha \in \mathbb{R}$.
- Dilation $d_\rho(z) = \rho z$, $\rho > 0$.
- Reflection in y-axis $\overline{r}(z) = -\overline{z}$.
- Inversion in the unit circle $I(z) = \frac{1}{\overline{z}}$.

***Remark.** Orientation-preserving $\mathbb{H}^2$-isometries are also called Möbius Transformations, and we may identify

$$\left(f(z) = \frac{az+b}{cz+d}\right) \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and compositions of Möbius Transformations can be done by matrix multiplication. Since $\det f = ad - bc = 1$, it follows that the inverse of f is

$$\left(f^{-1}(z) = \frac{dz-b}{-cz+a}\right) \longleftrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note that even if $ad - bc \neq 1$ (in particular, if $ad - bc \in \mathbb{R}_{>0}$), we still get isometries on $\mathbb{H}^2$, and they are the same isometry as if we normalized the matrix to get determinant 1. To normalize a matrix $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, we divide each term by $\sqrt{a'd' - b'c'}$. So

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \iff \left(\begin{bmatrix} \frac{a'}{\sqrt{a'd'-b'c'}} & \frac{b'}{\sqrt{a'd'-b'c'}} \\ \frac{c'}{\sqrt{a'd'-b'c'}} & \frac{d'}{\sqrt{a'd'-b'c'}} \end{bmatrix} =: \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right), \text{ with } ad - bc = 1.$$

Given an orientation-reversing isometry $\overline{f}(z) = \frac{-a\overline{z}+b}{-c\overline{z}+d}$ with $ad-bc=1$, we can find its inverse as follows: Define $g(z) := \frac{az+b}{cz+d}$, and consider the reflection across the y-axis, $r(z) = -\overline{z}$. Then $\overline{f} = g \circ r$, so $\left(\overline{f}\right)^{-1} = (g \circ r)^{-1} = r \circ g^{-1}$. By above, we know $g^{-1}(z) = \frac{dz-b}{-cz+a}$, and thus

$$\left(\overline{f}\right)^{-1}(z) = r\left(\frac{dz-b}{-cz+a}\right) = -\overline{\left(\frac{dz-b}{-cz+a}\right)} = -\frac{\overline{dz-b}}{\overline{-cz+a}} = \frac{-d\overline{z}+b}{-c\overline{z}+a}.$$

Essentially, we are swapping the main diagonal and also negating it; the other diagonal (containing $b, -c$) remains unchanged.

***Theorem.** These are the descriptions of $\mathbb{H}^2$-isometries:

- **Rotation:** Rotation about $P \in \mathbb{H}^2$ with angle θ can be written as the product of two reflections across $\mathbb{H}^2$-lines intersecting at P at an angle $\theta/2$. The formula for rotation about i by θ in $\mathbb{H}^2$ is
$$\begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}.$$
- **Limit Rotation:** Limit rotations are products of two reflections across $\mathbb{H}^2$-lines intersecting on $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. We may conjugate to move the intersection point to ∞, in which case the limit rotation is of the form $t_\alpha(z) = z + \alpha$. Limit rotations have one fixed point on $\partial\mathbb{H}^2$. A general isometry of this form will move a given point along its horocycle (a circle tangent to $\partial\mathbb{H}^2$ at the fixed point, and going through the given point).
- **Translation:** Translations are products of two reflections across parallel (disjoint) $\mathbb{H}^2$-lines. They may be conjugated to the translation along the vertical line $x = 0$ is $d_\rho(z) = \rho z$, and points move along lines of the form $y = kx$ for some $k \in \mathbb{R}$. These isometries have two fixed points on $\partial\mathbb{H}^2$. A general isometry of this form will move a given point along the circle through the two fixed points and the given point.
- **Reflections and Glide Reflections:** These are the orientation-reversing $\mathbb{H}^2$-isometries, which are either just a reflection, or compositions of the above three types with a reflection.

Theorem. The sum of angles of a hyperbolic triangle is less than π. In particular, the area of a triangle with angles α, β, γ is $\pi - \alpha - \beta - \gamma$. This formula also holds if any of α, β, γ is equal to 0.