MAT 248B: PROBLEM SET 1

DUE TO MONDAY APR 13

Abstract. This problem set corresponds to the first week of the course MAT-248B Spring 2026. It is due Monday Apr 13 at 3:00pm submitted via Gradescope.

Task: Solve one of the problems below and submit it through Gradescope by Monday Apr 13 at 3pm. Be rigorous and precise in writing your solutions.

Problem 1. Find a basis of the k-vector space of sections for the line bundles $T\mathbb{P}^1_k$ and $\Omega\mathbb{P}^1_k$, i.e. the tangent and cotangent bundles of $\mathbb{P}^1_k$.

Problem 2. Show that there is no surjective map $\pi : \Sigma_g \longrightarrow \Sigma_{g'}$ between two projective curves $\Sigma_g, \Sigma_{g'}$ of genus g, g', respectively, if $g' > g$.

Problem 3. Consider an algebraic curve $C_d \subseteq \mathbb{P}^2_k$ of degree d, e.g.

$$C_d = \{x^d + y^d + z^d = 0\}.$$

Compute the normal bundle of C_d inside $\mathbb{P}^2_k$.

Problem 4. Let $\pi : \mathrm{Bl}_0(\mathbb{A}^2_k) \to \mathbb{A}^2_k$ be the blow-up of the affine plane $\mathbb{A}^2_k$ along the origin $0 \in \mathbb{A}^2_k$. Find the normal bundle of the exceptional divisor $E \cong \mathbb{P}^1_k \subseteq \mathrm{Bl}_0(\mathbb{A}^2_k)$.

Problem 5. Give an instance of an algebraic curve C defined over k and points $p_1, p_2, p_3, q_1, q_2, q_3 \in C$ such that

$$\dim_k \mathrm{H}^0(C, \mathcal{O}_C(p_1 + p_2 + p_3)) \neq \dim_k \mathrm{H}^0(C, \mathcal{O}_C(q_1 + q_2 + q_3)).$$

Problem 6. Find an algebraic variety X and two vector bundles $\mathcal{E}_1 \to X$ and $\mathcal{E}_2 \to X$ such that $\mathcal{E}_1 \cong \mathcal{E}_2$ as smooth vector bundles, but $\mathcal{E}_1 \ncong \mathcal{E}_2$ as algebraic vector bundles.

Problem 7. Consider an algebraic variety X and the diagonal embedding $\Delta : X \to X \times X$, $\Delta(p) = (p, p)$. Show that the normal bundle of Δ is algebraically isomorphic to the tangent bundle TX.

Problem 8. Consider the embedding $\iota_3 : \mathbb{P}^1_k \longrightarrow \mathbb{P}^3_k$ given by the global sections of $\mathcal{O}_{\mathbb{P}^1_k}(3)$, i.e. $\iota_3([s : t]) = [s^3 : s^2t : st^2 : t^3]$. Show that the normal bundle of $\iota_3(\mathbb{P}^1_k)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(5) \oplus \mathcal{O}_{\mathbb{P}^1_k}(5)$.

Problem 9. Let $\mathcal{E} \to \mathbb{P}^1_k$ be a vector bundle. Show that there exist $a_1, \dots, a_r \in \mathbb{Z}$ such that

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r).$$

That is, any vector bundle over $\mathbb{P}^1_k$ splits as a sum of line bundles.