

MAT 141: PROBLEMS ON HYPERBOLIC GEOMETRY SET 1

ABSTRACT. This is a list of problems on hyperbolic geometry, complementing the problems in Problem Sets 4 and 5 and the lectures and discussions on the topic.

1. PROBLEMS ON HYPERBOLIC DISTANCES

Problem 1. For each of the following pairs of points $P, Q \in \mathbb{H}^2$, compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$:

- (1) $P = 3i$ and $Q = 6i$.
- (2) $P = 2i + 7$ and $Q = 5i + 7$.
- (3) $P = i$ and $Q = 1 + 2i$.
- (4) $P = i$ and $Q = \rho + i$ where $\rho \in \mathbb{R}$ is any given real number.

Problem 2. For each of the following paths γ from $P = i$ to $Q = 1 + i$, draw the image of the path γ in \mathbb{H}^2 and compute their hyperbolic lengths $\ell_{\mathbb{H}^2}(\gamma)$:

- (1) $\gamma(t) = t + i, t \in [0, 1]$.
- (2) $\gamma(\theta) = \frac{1}{2} + \frac{\sqrt{5}}{2}(\cos \theta + i \sin \theta)$, where $\theta \in [\arccos(-\frac{1}{\sqrt{5}}), \arccos(\frac{1}{\sqrt{5}})]$.
- (3) $\gamma(t) = t + i \left(4 \left(t - \frac{1}{2}\right)^2\right), t \in [0, 1]$.
- (4) Which of the above three paths has minimal hyperbolic length?

Problem 3 Consider the point $i \in \mathbb{H}^2$.

- (1) Show that the set of points in \mathbb{H}^2 at hyperbolic unit distance from i is

$$S_i := \{z \in \mathbb{H}^2 : |z - i \cosh(1)| = \sinh(1)\}.$$

- (2) Draw the set S_i qualitatively in \mathbb{H}^2 , explain why it is a circle, find its center and its radius.

2. PROBLEMS ON HYPERBOLIC LINES

Problem 4. Consider $1 + i \in \mathbb{H}^2$ and $L := \{z \in \mathbb{H}^2 : \operatorname{Re}(z) = 1\}$ a hyperbolic line containing it. Given the point $i \in \mathbb{H}^2$ outside L , show that there are infinitely many hyperbolic lines through i that are parallel to L .

Problem 5. Consider two points $P = iy_1$ and $Q = iy_2$ in \mathbb{H}^2 , $y_1, y_2 \in \mathbb{R}_{>0}$ and $y_1 < y_2$.

- (1) Show that the unique hyperbolic line $E(P, Q) \subseteq \mathbb{H}^2$ given as the set of equidistant points to P and Q is

$$E(P, Q) = \{z \in \mathbb{H}^2 : |z|^2 = y_1 y_2\}.$$

- (2) Describe explicitly, using a formula, a hyperbolic isometry $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ whose fixed point set coincides with $E(P, Q)$.

Problem 6. Consider the two hyperbolic lines $L_1 := \{z \in \mathbb{H}^2 : |z - 2| = 2\}$ and $L_2 := \{z \in \mathbb{H}^2 : |z + 2| = 2\}$. Let $\rho \in (0, 1]$ be a given real number.

- (1) Find the coordinates of the point $z_1(\rho) \in L_1$, given as the unique point in L_1 with x -coordinate equal to ρ . Similarly, find the coordinates of the point $z_2(\rho) \in L_2$ given as the unique point in L_2 with x -coordinate $-\rho$.
- (2) Show that the hyperbolic distance between $z_1(\rho)$ and $z_2(\rho)$ is $\ln\left(\frac{2+\sqrt{\rho}}{2-\sqrt{\rho}}\right)$.
- (3) Conclude that two hyperbolic lines sharing a common vertex in $\mathbb{R} \subseteq \partial\mathbb{H}^2$ get closer and closer to each other in Euclidean distance with a rate of $\sqrt{\rho}$ when $\rho \rightarrow 0$.

Problem 7. Consider the three hyperbolic lines $L_1 := \{z \in \mathbb{H}^2 : |z - 1| = 1\}$, $L_2 := \{z \in \mathbb{H}^2 : |z + 1| = 1\}$ and $L_3 := \{z \in \mathbb{H}^2 : |z| = 2\}$. Show that the area of the hyperbolic triangle bounded by L_1, L_2, L_3 is π .

3. PROBLEMS ON HYPERBOLIC ISOMETRIES

Problem 8. Consider the two maps $f_1, f_2 : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ given by

$$f_1(z) = -\bar{z}, \quad f_2(z) = \frac{1}{\bar{z}}$$

- (1) Show that f_1 and f_2 are orientation-reversing hyperbolic isometries.
- (2) Describe the fixed points of f_1 and the fixed points of f_2 .
- (3) Determine whether $f_1 \circ f_2$ equals $f_2 \circ f_1$ or not.
- (4) Find all the fixed points of $f_1 \circ f_2$.
- (5) Show that $f_1 \circ f_2$ is not the identity but $(f_1 \circ f_2)^2$ is the identity.

Problem 9. Consider the map

$$f : \mathbb{H}^2 \rightarrow \mathbb{H}^2, \quad f(z) = \frac{3\bar{z} + 5}{5\bar{z} + 3}.$$

- (1) Show that f is an orientation-reversing hyperbolic isometry.
- (2) Prove that f has no fixed points.
- (3) Find a hyperbolic line $L \subseteq \mathbb{H}^2$ invariant under f , i.e. such that $f(L) = L$.
- (4) Find an explicit formula for the unique hyperbolic isometry $g : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that $f \circ g$ is the identity.
- (5) Find the fixed points of g .

Problem 10. Consider the map

$$f : \mathbb{H}^2 \rightarrow \mathbb{H}^2, \quad f(z) = \frac{(\sqrt{3} - 1)z + 2}{-z + (\sqrt{3} + 1)}.$$

- (1) Show that $1 + i$ is the unique fixed point of f .
- (2) Show that f^6 is the identity.
- (3) Prove that f is a hyperbolic rotation centered at $1 + i$ and determine its angle.