MAT 141: SOLUTIONS TO PROBLEMS ON HYPERBOLIC GEOMETRY

ABSTRACT. This is a list of solved problems on hyperbolic geometry, complementing the problems in Problem Sets 4 and 5 and the lectures and discussions on the topic.

1. PROBLEMS ON HYPERBOLIC DISTANCES

Problem 1. For each of the following pairs of points $P, Q \in \mathbb{H}^2$, compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$:

(1) $P = 3i$ and $Q = 6i$.
(2) $P = 2i + 7$ and $Q = 5i + 7$.
(3) $P = i$ and $Q = 1 + 2i$.
(4) $P = i$ and $Q = \rho + i$ where $\rho \in \mathbb{R}$ is any given real number.

(1) We've seen in class that the hyperbolic distance between two points $(x, y_1), (x, y_2)$ with $y_1 < y_2$ is $\ln\left(\frac{y_2}{y_1}\right)$. Thus, $d_{\mathbb{H}^2}(P, Q) = \ln(2)$.

(2) Using the formula above, $d_{\mathbb{H}^2}(P, Q) = \ln\left(\frac{5}{2}\right)$.

(3) We first construct the geodesic through P, Q, then compute the distance along the geodesic segment from P to Q. The line segment $\overline{PQ}$ has slope 1, so its perpendicular bisector has slope -1. The perpendicular bisector passes through the midpoint $\frac{P+Q}{2} = \frac{1}{2} + \frac{3}{2}i$, so the equation of the perpendicular bisector is $(y - \frac{3}{2}) = -(x - \frac{1}{2})$. When the perpendicular bisector intersects the x-axis, we have $y = 0$, and hence $-\frac{3}{2} = -x + \frac{1}{2} \implies x = 2$, so the center of the geodesic is $(2, 0)$. The radius is the Euclidean distance from $(2, 0)$ to P or Q, so we compute $d_{\mathbb{E}}((2, 0), (0, 1)) = \sqrt{5}$, and thus the geodesic has equation $(x - 2)^2 + y^2 = 5$. To find the distance from P to Q, we use the formula $ds = \frac{\sqrt{1+(dy/dx)^2}}{y}dx$. We rewrite the geodesic equation to get $y = \sqrt{5 - (x - 2)^2}$, and $\frac{dy}{dx} = \frac{-(x-2)}{\sqrt{5-(2-x)^2}}$. From P to Q, we have $0 \leq x \leq 1$, thus

$$d_{\mathbb{H}^2}(P, Q) = \int_0^1 \frac{\sqrt{1 + \frac{(-(x-2))^2}{5-(x-2)^2}}}{\sqrt{5 - (x - 2)^2}} dx = \ldots = \int_0^1 \frac{\sqrt{5}}{5 - (2 - x)^2} dx.$$

You do not need to simplify further.

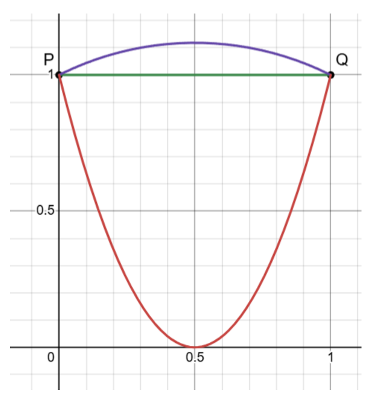
(4) We find the equation of the corresponding geodesic: The center is $\frac{\rho}{2}$, and the radius is $d_{\mathbb{E}}((\frac{\rho}{2}, 0), (0, 1)) = \sqrt{\frac{\rho^2}{4} + 1} = \sqrt{\frac{\rho^2+4}{4}}$, thus the geodesic is $(x - \frac{\rho}{2})^2 + y^2 = \frac{\rho^2+4}{4}$. We parametrize this by $(x(t), y(t)) = \left(\frac{\rho}{2} + \sqrt{\frac{\rho^2+4}{4}}\cos t, \sqrt{\frac{\rho^2+4}{4}}\sin t\right)$. For a general parametrization $(x(t), y(t)) = (\alpha + r\cos t, r\sin t)$, we have the formula $ds = \frac{\sqrt{(dx/dt)^2+(dy/dt)^2}}{y(t)}dt = \frac{\sqrt{r^2\sin^2 t+r^2\cos^2 t}}{r\sin t}dt = \frac{1}{\sin t}dt$. Now, for $\rho > 0$, we have the lower bound for t satisfies $\frac{\rho}{2} + \sqrt{\frac{\rho^2+4}{4}}\cos t = \rho \implies t = \cos^{-1}\left(\frac{\rho}{2} \cdot \sqrt{\frac{4}{\rho^2+4}}\right) = \cos^{-1}\left(\frac{\rho}{\sqrt{\rho^2+4}}\right) =: t_1$, and the upper bound satisfies $\frac{\rho}{2} + \sqrt{\frac{\rho^2+4}{4}}\cos t = 0 \implies t = \cos^{-1}\left(\frac{-\rho}{\sqrt{\rho^2+4}}\right) =: t_2$. Thus, $d_{\mathbb{H}^2}(P, Q) = \int_{t_1}^{t_2} \frac{1}{\sin t}dt = \ln(\tan(\frac{t}{2}))|_{t_1}^{t_2}$. Similarly, for

$\rho < 0$, we have $t_1 := \cos^{-1}\left(\frac{-\rho}{\sqrt{\rho^2+4}}\right)$, $t_2 := \cos^{-1}\left(\frac{\rho}{\sqrt{\rho^2+4}}\right)$, and the integral is the same one.

Problem 2. For each of the following paths γ from $P = i$ to $Q = 1 + i$, draw the image of the path γ in $\mathbb{H}^2$ and compute their hyperbolic lengths $\ell_{\mathbb{H}^2}(\gamma)$:

(1) $\gamma(t) = t + i$, $t \in [0, 1]$.
(2) $\gamma(\theta) = \frac{1}{2} + \frac{\sqrt{5}}{2}(\cos\theta + i\sin\theta)$, where $\theta \in [\arccos\left(-\frac{1}{\sqrt{5}}\right), \arccos\left(\frac{1}{\sqrt{5}}\right)]$.
(3) $\gamma(t) = t + i\left(4\left(t - \frac{1}{2}\right)^2\right)$, $t \in [0, 1]$.
(4) Which of the above three paths has minimal hyperbolic length?

(1) We have $x(t) = t, y(t) = 1, x'(t) = 1, y'(t) = 0$, from which we immediately compute $\ell_{\mathbb{H}^2}(\gamma) = \int_0^1 \frac{\sqrt{1^2+0^2}}{1} dt = \int_0^1 1 dt = 1$.
(2) We have $x(\theta) = \frac{1}{2} + \frac{\sqrt{5}}{2}\cos\theta, y(\theta) = \frac{\sqrt{5}}{2}\sin\theta$, and using the formula for computing the length of a geodesic segment, we have $\ell_{\mathbb{H}^2}(\gamma) = \ln\left(\tan\left(\frac{\theta}{2}\right)\right)\Big|_{\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)}^{\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}$.
(3) We have $x(t) = t, y(t) = 4(t - \frac{1}{2})^2$, and $t \in [0, 1]$. However, when $t = \frac{1}{2}$, we get $y = 0 \notin \mathbb{H}^2$. In particular, γ reaches a limit point, and hence it would follow that γ has length ∞.
(4) The second path has minimal hyperbolic length, since it is the geodesic segment from P to Q. Here is the picture of them (purple is (2),green is (1), red is (3)):



Problem 3 Consider the point $i \in \mathbb{H}^2$.

(1) Show that the set of points in $\mathbb{H}^2$ at hyperbolic unit distance from i is

$$S_i := \{z \in \mathbb{H}^2 : |z - i\cosh(1)| = \sinh(1)\}.$$

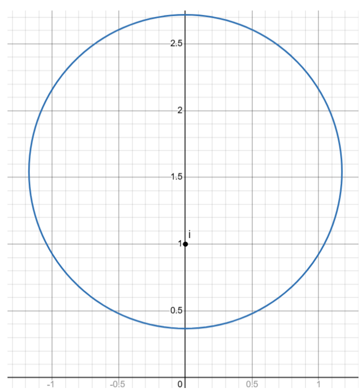
(2) Draw the set S_i qualitatively in $\mathbb{H}^2$, explain why it is a circle, find its center and its radius.

(1) We use the hyperbolic distance formula $d_{\mathbb{H}^2}(z,w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}\right)$, and set $w = i, z = x + yi \in \mathbb{H}^2$. Then, we have

$$\begin{aligned} d_{\mathbb{H}^2}(z,i) = 1 \iff 1 &= \cosh^{-1}\left(1 + \frac{|z-i|^2}{2\,\mathrm{Im}(z)}\right) = \cosh^{-1}\left(1 + \frac{|x+yi-i|^2}{2y}\right) \\ \iff& \cosh(1) = 1 + \frac{x^2 + (y-1)^2}{2y} \\ \iff& 2y(\cosh(1) - 1) = x^2 + (y-1)^2 = x^2 + y^2 - 2y + 1 \\ \iff& x^2 + y^2 - 2y\cosh(1) + 1 = 0 \\ \iff& x^2 + (y - \cosh(1))^2 - (\cosh(1))^2 + 1 = 0 \\ \iff& x^2 + (y - \cosh(1))^2 = (\cosh(1))^2 - 1 \end{aligned}$$

Using the identity $\cosh^2(x) - \sinh^2(x) = 1 \iff \sinh^2(x) = \cosh^2(x) - 1$, and noting that $x^2 + (y - \cosh(1))^2 = |x + i(y - \cosh(1))|^2 = |z - i\cosh(1)|^2$, we get $|z - i\cosh(1)|^2 = \sinh^2(1)$, and thus $|z - i\cosh(1)| = \sinh(1)$.

(2) Since $\cosh(1), \sinh(1)$ are constants, $|z - i\cosh(1)| = \sinh(1)$ precisely maps out a circle centered at $i\cosh(1) \approx 1.543i$ with radius $\sinh(1) \approx 1.175$.



2. Problems on hyperbolic lines

Problem 4. Consider $1 + i \in \mathbb{H}^2$ and $L := \{z \in \mathbb{H}^2 : \mathrm{Re}(z) = 1\}$ a hyperbolic line containing it. Given the point $i \in \mathbb{H}^2$ outside L, show that there are infinitely many hyperbolic lines through i that are parallel to L.

This is essentially the same problem as PSet 5 problem 2. See solution there.

Problem 5. Consider two points $P = iy_1$ and $Q = iy_2$ in $\mathbb{H}^2$, $y_1, y_2 \in \mathbb{R}_{>0}$ and $y_1 < y_2$.

(1) Show that the unique hyperbolic line $E(P,Q) \subseteq \mathbb{H}^2$ given as the set of equidistant points to P and Q is

$$E(P,Q) = \{z \in \mathbb{H}^2 : |z|^2 = y_1 y_2\}.$$

(2) Describe explicitly, using a formula, a hyperbolic isometry $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ whose fixed point set coincides with $E(P,Q)$.

(1) By a Theorem from class, we know that the set of equidistant points to two points is a hyperbolic line. For the case $P = iy_1, Q = iy_2$, it is clear by symmetry that the geodesic must be centered at 0. Then, we find the point on the imaginary axis equidistant to P, Q, and use this point to determine the radius. Suppose $z_0 = iy \in \mathbb{H}^2$ such that $d_{\mathbb{H}^2}(iy, iy_1) = d_{\mathbb{H}^2}(iy, iy_2)$, with $y_1 < y < y_2$. By a Theorem from class, we also know that the hyperbolic distance between two points $(x, y_1), (x, y_2)$ with the same real component is $\ln\left(\frac{y_2}{y_1}\right)$. Thus, we have $\ln\left(\frac{y}{y_1}\right) = \ln\left(\frac{y_2}{y}\right) \iff \frac{y}{y_1} = \frac{y_2}{y} \iff y^2 = y_1 y_2 \iff y = \sqrt{y_1 y_2}$. In particular, the Euclidean distance from 0 to z_0 is $y = \sqrt{y_1 y_2}$, and this is the radius of the geodesic. Therefore, the set of equidistant points to P, Q is the geodesic centered at 0 with radius $\sqrt{y_1 y_2}$, and this is precisely $E(P, Q)$.

(2) We want to find the formula for reflection across $E(P, Q)$. Recall the standard inversion across the unit circle $|z| = 1$ is $I(z) = \frac{1}{\bar{z}}$, so we do a conjugation by $d_{\sqrt{y_1 y_2}}(z) = \sqrt{y_1 y_2} \cdot z, d^{-1}_{\sqrt{y_1 y_2}}(z) = \frac{z}{\sqrt{y_1 y_2}}$ ($d^{-1}_{\sqrt{y_1 y_2}}$ scales $E(P, Q)$ to align with the unit circle, and $d_{\sqrt{y_1 y_2}}$ scales the unit circle to align with $E(P, Q)$), and therefore the reflection across $E(P, Q)$ is

$$r_E(z) = d_{\sqrt{y_1 y_2}} \circ I \circ d^{-1}_{\sqrt{y_1 y_2}}(z) = d_{\sqrt{y_1 y_2}} \circ I\left(\frac{z}{\sqrt{y_1 y_2}}\right) = d_{\sqrt{y_1 y_2}}\left(\frac{\sqrt{y_1 y_2}}{\bar{z}}\right) = \frac{y_1 y_2}{\bar{z}}.$$

Problem 6. Consider the two hyperbolic lines $L_1 := \{z \in \mathbb{H}^2 : |z - 2| = 2\}$ and $L_2 := \{z \in \mathbb{H}^2 : |z + 2| = 2\}$. Let $\rho \in (0, 1]$ be a given real number.

(1) Find the coordinates of the point $z_1(\rho) \in L_1$, given as the unique point in L_1 with x-coordinate equal to ρ. Similarly, find the coordinates of the point $z_2(\rho) \in L_2$ given as the unique point in L_2 with x-coordinate $-\rho$.

(2) Show that the hyperbolic distance between $z_1(\rho)$ and $z_2(\rho)$ is $\ln\left(\frac{2+\sqrt{\rho}}{2-\sqrt{\rho}}\right)$.

(3) Conclude that two hyperbolic lines sharing a common vertex in $\mathbb{R} \subseteq \partial\overline{\mathbb{H}}^2$ get closer and closer to each other in Euclidean distance with a rate of $\sqrt{\rho}$ when $\rho \to 0$.

(1) $L_1 = \{(x, y) \in \mathbb{H}^2 : (x - 2)^2 + y^2 = 4\}$, so $z_1(\rho) = \rho + yi$ such that $(\rho - 2)^2 + y^2 = 4 \implies y = \sqrt{4 - (\rho - 2)^2}$. Similarly for $z_2(\rho)$, we get $z_2(\rho) = -\rho + yi$ such that $(2 - \rho)^2 + y^2 = 4 \implies y = \sqrt{4 - (\rho - 2)^2}$. Thus, $z_1(\rho) = \rho + i\sqrt{4 - (\rho - 2)^2}, z_2(\rho) = -\rho + i\sqrt{4 - (\rho - 2)^2}$.

(2) First, we find the general formula for distance between $(\alpha, \beta), (-\alpha, \beta)$ for $\alpha, \beta > 0$. The circle geodesic through them is centered at $(0, 0)$ with radius $\sqrt{\alpha^2 + \beta^2}$, so the equation is $x^2 + y^2 = \alpha^2 + \beta^2$. Then, we have $y = \sqrt{\alpha^2 + \beta^2 - x^2}$, $\frac{dy}{dx} = \frac{-x}{\sqrt{\alpha^2+\beta^2-x^2}}$, and thus

$$d_{\mathbb{H}^2}((\alpha, \beta), (-\alpha, \beta)) = \int_{-\alpha}^{\alpha} \frac{\sqrt{1 + \frac{x^2}{\alpha^2+\beta^2-x^2}}}{\sqrt{\alpha^2 + \beta^2 - x^2}} dx = \int_{-\alpha}^{\alpha} \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha^2 + \beta^2 - x^2} dx.$$

Therefore, $d_{\mathbb{H}^2}(z_1(\rho), z_2(\rho)) = \int_{-\rho}^{\rho} \frac{\sqrt{\rho^2+4-(\rho-2)^2}}{\rho^2+4-(\rho-2)^2-x^2} dx = \int_{-\rho}^{\rho} \frac{\sqrt{4\rho}}{4\rho-x^2} dx = \int_{-\rho}^{\rho} \frac{2\sqrt{\rho}}{4\rho-x^2} dx$. Then, we apply partial fractions:

$$\int_{-\rho}^{\rho} \frac{2\sqrt{\rho}}{4\rho - x^2} dx = \int_{-\rho}^{\rho} \frac{A}{2\sqrt{\rho} - x} + \frac{B}{2\sqrt{\rho} + x} dx$$

$$\implies 2\sqrt{\rho} = 2\sqrt{\rho}A + Ax + 2\sqrt{\rho}B - Bx \implies A = B = \frac{1}{2}$$

$$\begin{aligned}\int_{-\rho}^{\rho} \frac{2\sqrt{\rho}}{4\rho - x^2} dx &= \int_{-\rho}^{\rho} \frac{1/2}{2\sqrt{\rho} - x} + \frac{1/2}{2\sqrt{\rho} + x} dx \\ &= \left(\frac{1}{2}(-\ln(2\sqrt{\rho} - x) + \ln(2\sqrt{\rho} + x)\right)\Big|_{-\rho}^{\rho} \\ &= \left(\frac{1}{2}\ln\left(\frac{2\sqrt{\rho} + x}{2\sqrt{\rho} - x}\right)\right)\Big|_{-\rho}^{\rho} \\ &= \frac{1}{2}\left(\ln\left(\frac{2\sqrt{\rho} + \rho}{2\sqrt{\rho} - \rho}\right) - \ln\left(\frac{2\sqrt{\rho} - \rho}{2\sqrt{\rho} + \rho}\right)\right) \\ &= \frac{1}{2}\left(\ln\left(\frac{2\sqrt{\rho} + \rho}{2\sqrt{\rho} - \rho}\right) + \ln\left(\frac{2\sqrt{\rho} + \rho}{2\sqrt{\rho} - \rho}\right)\right) \\ &= \ln\left(\frac{2\sqrt{\rho} + \rho}{2\sqrt{\rho} - \rho}\right) = \ln\left(\frac{2 + \sqrt{\rho}}{2 - \sqrt{\rho}}\right),\end{aligned}$$

as desired.

(3) We know that the limit

$$\lim_{\rho \to 0} \ln\left(\frac{2 + \sqrt{\rho}}{2 - \sqrt{\rho}}\right) = 0$$

vanishes. The question is asking how fast it vanishes, i.e. the rate at which it converges to 0. To compute that rate, and show it is equal to $\sqrt{\rho}$, we compute the Taylor (a.k.a. Maclaurin) series expansion expansion of $\ln\left(\frac{2+\sqrt{\rho}}{2-\sqrt{\rho}}\right)$ around $\rho = 0$. Since the Taylor series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$ is

$$\ln\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right),$$

we re-write

$$\frac{2 + \sqrt{\rho}}{2 - \sqrt{\rho}} = \frac{2\left(1 + \frac{\sqrt{\rho}}{2}\right)}{2\left(1 - \frac{\sqrt{\rho}}{2}\right)} = \frac{1 + \frac{\sqrt{\rho}}{2}}{1 - \frac{\sqrt{\rho}}{2}}.$$

Therefore we obtain

$$\ln\left(\frac{1 + \frac{\sqrt{\rho}}{2}}{1 - \frac{\sqrt{\rho}}{2}}\right) = 2\left(\left(\frac{\sqrt{\rho}}{2}\right) + \frac{1}{3}\left(\frac{\sqrt{\rho}}{2}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{\rho}}{2}\right)^5 + \frac{1}{7}\left(\frac{\sqrt{\rho}}{2}\right)^7 + \dots\right)$$

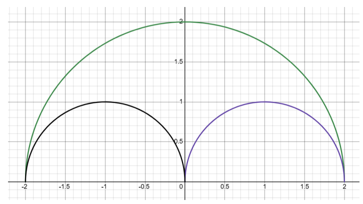
This expression can be simplified to

$$\begin{aligned}\ln\left(\frac{1 + \frac{\sqrt{\rho}}{2}}{1 - \frac{\sqrt{\rho}}{2}}\right) &= 2\left(\frac{\rho^{1/2}}{2} + \frac{1}{3}\cdot\frac{\rho^{3/2}}{8} + \frac{1}{5}\cdot\frac{\rho^{5/2}}{32} + \frac{1}{7}\cdot\frac{\rho^{7/2}}{128} + \dots\right) \\ &= \rho^{1/2} + \frac{2}{24}\rho^{3/2} + \frac{2}{160}\rho^{5/2} + \frac{2}{896}\rho^{7/2} + \dots \\ &= \sqrt{\rho} + \frac{1}{12}\rho^{3/2} + \frac{1}{80}\rho^{5/2} + \frac{1}{448}\rho^{7/2} + \mathcal{O}(\rho^{9/2}).\end{aligned}$$

Thus the convergence rate near $\rho \to 0$ is indeed dominated by $\sqrt{\rho}$, as required.

Problem 7. Consider the three hyperbolic lines $L_1 := \{z \in \mathbb{H}^2 : |z-1| = 1\}$, $L_2 := \{z \in \mathbb{H}^2 : |z+1| = 1\}$ and $L_3 := \{z \in \mathbb{H}^2 : |z| = 2\}$. Show that the area of the hyperbolic triangle bounded by L_1, L_2, L_3 is π.

A Theorem from class states that the area of a hyperbolic triangle with angles α, β, γ is $\pi - \alpha - \beta - \gamma$. So to show this triangle has area π, it suffices to show that the angles α, β, γ are all equal to 0, and this happens precisely when the lines meet on $\partial\mathbb{H}^2$ (this is because the angle a geodesic makes with $\partial\mathbb{H}^2$ is always $\pi/2$ by construction, so when two different geodesics meet at $\partial\mathbb{H}^2$, they both make an angle $\pi/2$ with $\partial\mathbb{H}^2$, and hence the angle between the two geodesics must be 0). Note that L_1 has endpoints $0, 2$, L_2 has endpoints $-2, 0$, and L_3 has endpoints $2, -2$, and therefore the angles are all equal to 0, hence the area of this triangle is π. Here's what the triangle looks like:



3. Problems on hyperbolic isometries

Problem 8. Consider the two maps $f_1, f_2 : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ given by

$$f_1(z) = -\overline{z}, \quad f_2(z) = \frac{1}{\overline{z}}$$

(1) Show that f_1 and f_2 are orientation-reversing hyperbolic isometries.

(2) Describe the fixed points of f_1 and the fixed points of f_2.

(3) Determine whether $f_1 \circ f_2$ equals $f_2 \circ f_1$ or not.

(4) Find all the fixed points of $f_1 \circ f_2$.

(5) Show that $f_1 \circ f_2$ is not the identity but $(f_1 \circ f_2)^2$ is the identity.

(1) It suffices to write f_1, f_2 in the form $f(z) = \frac{-a\overline{z}+b}{-c\overline{z}+d}$, with $ad - bc = 1$, since we know from a theorem that the orientation reversing isometries on $\mathbb{H}^2$ are of this form. So, we have $f_1(z) = \frac{-\overline{z}+0}{-0\overline{z}+1}$, $f_2(z) = \frac{-0\overline{z}+1}{-(-1)\overline{z}+0}$, and these already satisfy $ad-bc = 1$, hence they are orientation-reversing hyperbolic isometries.
(2) We've seen in discussion (or maybe also in class) that $f_1(z) = -\overline{z}$ is the reflection across the y-axis, hence the fixed points of f_1 are all the points on the y-axis. We've also seen in discussion that $f_2(z)$ is the inversion across the unit circle, hence the fixed points of f_2 are all the points on the unit circle $|z| = 1$.
(3) Note that the y-axis and the unit circle intersect at an angle of $\pi/2$, thus $f_1 \circ f_2$ and $f_2 \circ f_1$ are both rotations by π. We can also do this with explicit computations:

$$f_1 \circ f_2(z) = f_1\left(\frac{1}{\overline{z}}\right) = f_1\left(\frac{z}{|z|^2}\right) = -\frac{\overline{z}}{|z|^2} = -\frac{\overline{z}}{\overline{z}z} = -\frac{1}{z}$$

$$f_2 \circ f_1(z) = f_2(-\overline{z}) = \frac{1}{\overline{-\overline{z}}} = -\frac{1}{z}.$$

(4) $f_1 \circ f_2$ is the rotation about i by π, hence there is a single fixed point of $f_1 \circ f_2$, namely i.

(5) Since $f_1 \circ f_2(\frac{i}{2}) = -\frac{1}{i/2} = -\frac{2}{i} = 2i \neq \frac{i}{2}$, it follows that $f_1 \circ f_2 \neq \text{Id}$. However, we have $(f_1 \circ f_2)^2(z) = (f_1 \circ f_2)(-\frac{1}{z}) = -\frac{1}{-1/z} = z$, hence $(f_1 \circ f_2)^2 = \text{Id}$.

Problem 9. Consider the map

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{3\overline{z}+5}{5\overline{z}+3}.$$

(1) Show that f is an orientation-reversing hyperbolic isometry.

(2) Prove that f has no fixed points.

(3) Find a hyperbolic line $L \subseteq \mathbb{H}^2$ invariant under f, i.e. such that $f(L) = L$.

(4) Find an explicit formula for the unique hyperbolic isometry $g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ such that $f \circ g$ is the identity.

(5) Find the fixed points of g.

(1) As with problem 8, we want to write f in the form $f(z) = \frac{-a\overline{z}+b}{-c\overline{z}+d}$, with $ad - bc = 1$, and then it would follow from a theorem in class that f is orientation-reversing. For $f(z) = \frac{3\overline{z}+5}{5\overline{z}+3}$, we have $ad - bc = -9 - (-25) = 16$, and we want to normalize by dividing by $\sqrt{16} = 4$. So $f(z) = \frac{\frac{3}{4}\overline{z}+\frac{5}{4}}{\frac{5}{4}\overline{z}+\frac{3}{4}}$, and here $ad - bc = 1$, hence it is an orientation-reversing hyperbolic isometry.

(2) Assume $z = x + yi \in \mathbb{H}^2$ is fixed by f. Then

$$\begin{aligned} x + yi = z = \frac{3\overline{z}+5}{5\overline{z}+3} &= \frac{3x+5-3yi}{5x+3-5yi} \\ \iff &(x+yi)(5x+3-5yi) = 3x+5-3yi \\ \iff &5x^2 + 3x - 5xyi + 5xyi + 3yi + 5y^2 = 3x + 5 - 3yi \\ \iff &5x^2 + 5y^2 = 5 \text{ and } 3yi = -3yi, \end{aligned}$$

But the latter implies $y = 0$, hence $z \notin \mathbb{H}^2$. So f has no fixed points.

(3) To find a hyperbolic line invariant under f, we look for the two endpoints of the line on $\partial\mathbb{H}^2$. If $x \in \mathbb{R} \subseteq \partial\mathbb{H}^2$ is fixed by f, then we have $x = \frac{3x+5}{5x+3} \implies 5x^2 + 3x = 3x + 5 \implies x^2 = 1 \implies x = \pm 1$. So the geodesic through $1, -1$ is invariant under f, i.e. the unit circle $|z| = 1$.

(4) We find the inverse of $f(z) = \frac{\frac{3}{4}\overline{z}+\frac{5}{4}}{\frac{5}{4}\overline{z}+\frac{3}{4}}$. Define $\widehat{f}(z) := \frac{-\frac{3}{4}z+\frac{5}{4}}{-\frac{5}{4}z+\frac{3}{4}}$, and consider the reflection across the y-axis $r(z) = -\overline{z}$. We have that $f = \widehat{f} \circ r$, and thus $g = f^{-1} = (\widehat{f} \circ r)^{-1} = r \circ \widehat{f}^{-1}$. Using matrix representations, $\widehat{f} \longleftrightarrow \begin{bmatrix} -\frac{3}{4} & \frac{5}{4} \\ -\frac{5}{4} & \frac{3}{4} \end{bmatrix}$, and its inverse matrix is $\begin{bmatrix} \frac{3}{4} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{3}{4} \end{bmatrix} \longleftrightarrow \widehat{f}^{-1}(z) := \frac{\frac{3}{4}z-\frac{5}{4}}{\frac{5}{4}z-\frac{3}{4}}$. Then, we compute

$$g(z) = r \circ \widehat{f}^{-1}(z) = -\overline{\left(\frac{\frac{3}{4}z-\frac{5}{4}}{\frac{5}{4}z-\frac{3}{4}}\right)} = -\frac{\overline{\frac{3}{4}z-\frac{5}{4}}}{\overline{\frac{5}{4}z-\frac{3}{4}}} = -\frac{\frac{3}{4}\overline{z}-\frac{5}{4}}{\frac{5}{4}\overline{z}-\frac{3}{4}} = \frac{-\frac{3}{4}\overline{z}+\frac{5}{4}}{\frac{5}{4}\overline{z}-\frac{3}{4}}.$$

(5) If z_0 is fixed by g, then $f \circ g(z_0) = f(z_0) \neq z_0$ because we showed in part (2) that f has no fixed points. But by part (4), $f \circ g = \text{Id}$, thus we must have $f \circ g(z_0) = z_0$, which is a contradiction. Therefore, g has no fixed points as well. Alternatively, we can also compute explicitly for fixed points in the same way as part (2).

Problem 10. Consider the map

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{(\sqrt{3}-1)z+2}{-z+(\sqrt{3}+1)}.$$

(1) Show that $1+i$ is the unique fixed point of f.

(2) Show that f^6 is the identity.

(3) Prove that f is a hyperbolic rotation centered at $1+i$ and determine its angle.

(1) Suppose $f(z_0) = z_0$. Then we have

$$\begin{aligned} z &= \frac{(\sqrt{3}-1)z+2}{-z+(\sqrt{3}+1)} \\ &\iff -z^2+\sqrt{3}z+z = \sqrt{3}z - z + 2 \\ &\iff z^2 - 2z + 2 = 0 \\ &\iff z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i, \end{aligned}$$

and since $1 - i \notin \mathbb{H}^2$, it follows that the unique fixed point of f is $1+i$.

(2) We have $f \longleftrightarrow \begin{bmatrix} \sqrt{3}-1 & 2 \\ -1 & \sqrt{3}+1 \end{bmatrix}$, and we just compute f^6 directly:

$$\begin{aligned} \begin{bmatrix} \sqrt{3}-1 & 2 \\ -1 & \sqrt{3}+1 \end{bmatrix}^6 &= \left(\begin{bmatrix} \sqrt{3}-1 & 2 \\ -1 & \sqrt{3}+1 \end{bmatrix}^3 \right)^2 \\ &= \left(\begin{bmatrix} \sqrt{3}-1 & 2 \\ -1 & \sqrt{3}+1 \end{bmatrix} \begin{bmatrix} 3-2\sqrt{3}+1-2 & 2\sqrt{3}-2+2\sqrt{3}+2 \\ -\sqrt{3}+1-\sqrt{3}-1 & -2+3+2\sqrt{3}+1 \end{bmatrix} \right)^2 \\ &= \left(\begin{bmatrix} \sqrt{3}-1 & 2 \\ -1 & \sqrt{3}+1 \end{bmatrix} \begin{bmatrix} 2-2\sqrt{3} & 4\sqrt{3} \\ -2\sqrt{3} & 2+2\sqrt{3} \end{bmatrix} \right)^2 \\ &= \left(\begin{bmatrix} 2\sqrt{3}-6-2+2\sqrt{3}-4\sqrt{3} & 12-4\sqrt{3}+4+4\sqrt{3} \\ -2+2\sqrt{3}-6-2\sqrt{3} & -4\sqrt{3}+2\sqrt{3}+6+2+2\sqrt{3} \end{bmatrix} \right)^2 \\ &= \left(\begin{bmatrix} -8 & 16 \\ -8 & 8 \end{bmatrix} \right)^2 = \left(8 \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \right)^2 \\ &= 64 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned}$$

which we can just normalize to get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence, $f^6 = \mathrm{Id}$.

(3) Since f is an orientation-preserving isometry, and fixes a single point in $\mathbb{H}^2$, it follows from the classification of Möbius Transformations that f is a rotation about its fixed point $1+i$. To determine the angle of rotation, we consider the geodesic $L := \{(x,y) \in \mathbb{H}^2 : x = 1\}$, and observe how it behaves under f. Note that $f(1) = \frac{\sqrt{3}+1}{-1+\sqrt{3}+1} = \frac{3+\sqrt{3}}{3} = 1 + \frac{\sqrt{3}}{3}$, and $f(\infty) = \frac{\sqrt{3}-1}{-1} = 1 - \sqrt{3}$. So $f(L)$ is the geodesic with endpoints at $1+\frac{\sqrt{3}}{3}, 1-\sqrt{3}$, center $\frac{1+\frac{\sqrt{3}}{3}+1-\sqrt{3}}{2} = \frac{2-\frac{2\sqrt{3}}{3}}{2} = 1 - \frac{\sqrt{3}}{3}$ and radius $\frac{1+\frac{\sqrt{3}}{3}-1+\sqrt{3}}{2} = \frac{2\sqrt{3}}{3}$, so its equation is

$$\left\{ (x,y) \in \mathbb{H}^2 : \left(x - 1 + \frac{\sqrt{3}}{3} \right)^2 + y^2 = \frac{4}{3} \right\}.$$

We want to find the equation of the tangent line at $1+i$, and find the angle between this line and the line $x=1$. We consider the vector $\frac{\sqrt{3}}{3}, 1\rangle$ (this is the vector from the center of the circle to $1+i$), and note that the tangent line is perpendicular to this vector, hence for any point (x, y) on the tangent line, we must have $\langle \frac{\sqrt{3}}{3}, 1\rangle \cdot \langle x - 1, y-1\rangle = 0 \iff \frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{3} + y - 1 = 0 \iff y = -\frac{\sqrt{3}}{3}x + 1 + \frac{\sqrt{3}}{3}$. Consider the Euclidean right triangle bounded by L, the tangent line, and the x-axis. We note that the tangent line intersects the x-axis at $(\sqrt{3}+1, 0)$, and that the other two vertices are $(1,0), (1,1)$. This forms a special triangle $(30^\circ - 60^\circ - 90^\circ)$ triangle, and thus the angle formed by L and the tangent line is $\frac{\pi}{3}$. It follows that its vertical angle is also $\frac{\pi}{3}$, and hence f is a rotation about $1+i$ by angle $\frac{\pi}{3}$.

Alternatively, we could also do a conjugation by the limit rotation $t_1(z) = z+1$ and look at $t_1^{-1} \circ f \circ t_1$, which is a rotation about i. We compute this formula directly:

$$\begin{aligned} t_1^{-1} \circ f \circ t_1(z) &= t_1^{-1} \circ f(z+1) \\ &= t_1^{-1}\left(\frac{(\sqrt{3}-1)z + \sqrt{3} - 1 + 2}{-z - 1 + \sqrt{3} + 1}\right) \\ &= \frac{(\sqrt{3}-1)z + \sqrt{3} + 1}{-z + \sqrt{3}} - \frac{-z+\sqrt{3}}{-z+\sqrt{3}} \\ &= \frac{\sqrt{3}z + 1}{-z + \sqrt{3}} \longleftrightarrow \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}, \end{aligned}$$

which has determinant 4, and we can normalize to get $t_1^{-1} \circ f \circ t_1 \longleftrightarrow \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. Recall we've seen in the homework that the rotation about i by angle θ is given by the matrix $\begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{bmatrix}$. By observation, we have that $\cos\pi/6 = \frac{\sqrt{3}}{2}, \sin\pi/6 = \frac{1}{2}$, and hence $t_1^{-1} \circ f \circ t_1 \longleftrightarrow \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos\pi/6 & \sin\pi/6 \\ -\sin\pi/6 & \cos\pi/6 \end{bmatrix}$, and therefore $t_1^{-1} \circ f \circ t_1$ is the rotation about i by angle $\pi/3$. We conclude that f is the rotation about $1+i$ by angle $\pi/3$.