University of California Davis **Name (Print):** ____________________
Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Midterm Examination **May 1 2026**
Time Limit: 50 Minutes

This examination document contains 8 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Let $P = (0,0), Q = (1,0), R = (0,3), S = (-3,0) \in \mathbb{R}^2$.

   (a) (10 points) Find the distances $d(P,S), d(Q,S), d(R,S)$.

   (b) (10 points) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = (0,2)$, $f(Q) = (-1,2)$ and $f(R) = (0,5)$. Find the image $f(S)$ of the point $S \in \mathbb{R}^2$.

2. (20 points) Consider the points $O = (0,0), P = (1,0), Q = (1,1) \in \mathbb{R}^2$ in the Euclidean plane. Let $L, M \subseteq \mathbb{R}^2$ be the unique lines such that $O, P \in L$ and $O, Q \in M$.

   (a) (10 points) Describe the fixed points of the isometry $\overline{r}_M \circ \overline{r}_L$.

   (b) (5 points) Show that the isometry given by applying $\overline{r}_M \circ \overline{r}_L$ four times is the identity. That is, show that the composition

   $$(\overline{r}_M \circ \overline{r}_L)^4 = \overline{r}_M \circ \overline{r}_L \circ \overline{r}_M \circ \overline{r}_L \circ \overline{r}_M \circ \overline{r}_L \circ \overline{r}_M \circ \overline{r}_L$$

   is the identity.

   (c) (5 points) Let $N \subseteq \mathbb{R}^2$ be the unique line containing $P, Q \in \mathbb{R}^2$. Show that the isometry $\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L$ is a rotation.

3. (20 points) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

   (a) (6 points) Draw and describe the Γ-orbits of the points $P = (1, -2), Q = (-0.1, 0.9) \in \mathbb{R}^2$.

   (b) (6 points) Find the distance $d(\Gamma P, \Gamma Q)$ of $\Gamma P, \Gamma Q \in T^2$ in the 2-torus.

(c) (4 points) Give three different lines $L_0, L_1, L_2 \subseteq T^2$ in the 2-torus such that both points $\Gamma P, \Gamma Q \in T^2$ belong to each L_0, L_1 and L_2.

(d) (4 points) Find the number of intersection points between the two lines

$$\{(x, y) \in T^2 : x = y\} \subseteq T^2 \quad \text{and} \quad \{(x, y) \in T^2 : x = 0.3\} \subseteq T^2.$$

4. (20 points) Consider the 2-sphere

$$S^2 := \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance, and the points

$$P = (1,0,0), \quad Q = (0,1,0), \quad R = (0,0,1), \quad S = \frac{1}{\sqrt{2}}(0,1,1)$$

in S^2. Solve the following parts:

(a) (6 points) Compute the three spherical distances $d_{S^2}(P,S)$, $d_{S^2}(Q,S)$ and $d_{S^2}(R,S)$.

(b) (6 points) Let $E \subseteq S^2$ be the unique line through P, Q, and $L \subseteq S^2$ the unique line through P, S. Describe the image of R under the isometries $r_E \circ r_L$ and $r_L \circ r_E$.

(c) (4 points) Find the fixed points of the isometries $r_E \circ r_L$ and $r_L \circ r_E$.

(d) (4 points) Describe the unique line $M \subseteq S^2$ equidistant to S and Q.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

   (a) (2 points) Any isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ with no fixed points must be a translation.

   (1) True. (2) False.

   (b) (2 points) The composition of two rotations in $\mathbb{R}^2$ is a rotation.

   (1) True. (2) False.

   (c) (2 points) There are no lines $L, N \subseteq M$ in the twisted cylinder with $|L \cap N| = 2$.

   (1) True. (2) False.

   (d) (2 points) An isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is different from the identity cannot have more than three fixed points.

   (1) True. (2) False.

   (e) (2 points) In the Klein bottle, two non-parallel lines intersect either once, twice or infinitely many times.

   (1) True. (2) False.

   (f) (2 points) Any orientation-preserving isometry of $\mathbb{R}^2$ can be written as the composition of two reflections.

   (1) True. (2) False.

   (g) (2 points) Any composition of an even number of reflections, including zero, can be expressed as a composition of two reflections.

   (1) True. (2) False.

   (h) (2 points) For any pair of points $P, Q \in C$ in the cylinder, there are infinitely many distinct lines $L \subseteq C$ containing $P, Q \in K$.

   (1) True. (2) False.

   (i) (2 points) Let $L, M \subseteq S^2$ be two different spherical lines. Then $\overline{r}_L \circ \overline{r}_M = \overline{r}_M \circ \overline{r}_L$.

   (1) True. (2) False.

   (j) (2 points) Any isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ is uniquely determined by the images $f(0,0), f(1,2)$ and $f(-4,-8)$ of the three points $(0,0), (1,2)$ and $(-4,-8)$.

   (1) True. (2) False.