MAT 141: PROBLEM SET 1

DUE TO FRIDAY APR 10 AT 11:00AM

Abstract. This is the first problem set for the Euclidean and Non-Euclidean Geometry Course in the Spring Quarter 2026, via online Gradescope submission.

Purpose: The goal of this assignment is to practice problems on the geometry of the Euclidean Plane $\mathbb{R}^2$. In particular, we would like to become familiar with the distance function, isometries and composition of isometries.

Task: Solve Problems 1 through 7 below. Problems 2, 3, 4, and 7 will be graded. Problems 1, 5, and 6 will not be graded but I trust that you will work on them.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. Though it is not recommended, if you do use an AI assistant, please clearly acknowledge that in the upper-left corner of the pages of each problem.

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. (ungraded) Consider the Euclidean distance in $\mathbb{R}^2$, i.e. the distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(i) Prove that this distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following three properties:

(a) For any pairs of points $P, Q \in \mathbb{R}^2$,

$$d(P, Q) \geq 0,$$

and equality only occurs if $P = Q$.

(b) For any pairs of points $P, Q \in \mathbb{R}^2$,

$$d(P, Q) = d(Q, P).$$

(c) For any three points $P, Q, R \in \mathbb{R}^2$,

$$d(P,Q) \le d(Q,R) + d(R,P).$$

(ii) Describe for which triples of points $P, Q, R \in \mathbb{R}^2$ the general inequality

$$d(P,Q) \le d(Q,R) + d(R,P).$$

that you have proven in Part (i).c is actually an equality.

(i) (a) Clearly, the function $f(x) = \sqrt{x}$ is always greater than or equal to 0, hence $d(P,Q) \ge 0$ for any pair $P, Q \in \mathbb{R}^2$. $0 = d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \iff (x_2 - x_1) = 0, (y_2 - y_1) = 0 \iff x_1 = x_2, y_1 = y_2 \iff P = Q$, so equality only occurs if $P = Q$.

(b) Since $(a-b)^2 = (-(b-a))^2 = (-1)^2(b-a)^2 = (b-a)^2$ for any $a, b \in \mathbb{R}$, it follows that $d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d(Q,P)$.

(c) This is proven in the textbook, at the end of section 1.3.

(ii) We get equality if and only if R is on the segment $\overline{PQ}$. To prove this, we first assume R is on the segment $\overline{PQ}$. Without loss of generality, assume $P = (0,0), Q = (x,0), R = (\alpha,0)$, where $0 \le \alpha \le x$. Then $d(Q,R) + d(R,P) = \sqrt{(x-\alpha)^2} + \sqrt{\alpha^2} = x - \alpha + \alpha = x = \sqrt{x^2} = d(P,Q)$. Conversely, assume R is not on the segment $\overline{PQ}$, so let $R = (\alpha, \beta)$, with $\beta \ne 0$. We continue to use the assumption that $P = (0,0), Q = (x,0)$. Then we may drop a perpendicular from R to the x-axis, yielding the point $S := (\alpha, 0)$. Then we get two right triangles $\triangle PSR, \triangle QSR$. By the Pythagorean Theorem, $d(R,P)^2 = d(P,S)^2 + d(S,R)^2, d(Q,R)^2 = d(Q,S)^2 + d(S,R)^2$. So $d(Q,R) + d(R,P) = \sqrt{d(Q,S)^2 + d(S,R)^2} + \sqrt{d(P,S)^2 + d(S,R)^2}$. We want to show $\sqrt{d(Q,S)^2 + d(S,R)^2} + \sqrt{d(P,S)^2 + d(S,R)^2} > d(P,Q)$. If $\alpha < 0$ or $\alpha > x$, then $d(Q,S) > d(P,Q)$ or $d(P,S) > d(P,Q)$ respectively, and the inequality follows immediately. If $0 \le \alpha \le x$, then S is on the segment $\overline{PQ}$, and we've proved already that $d(P,Q) = d(P,S) + d(S,Q)$. Now, since $d(S,R) > 0$, we observe that $\sqrt{d(Q,S)^2 + d(S,R)^2} + \sqrt{d(P,S)^2 + d(S,R)^2} > \sqrt{d(Q,S)^2} + \sqrt{d(P,S)^2} = d(Q,S) + d(P,S) = d(P,Q)$, as desired.

Problem 2. (25 points) For each of the following maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, decide whether they are isometries of the Euclidean plane $\mathbb{R}^2$ or not. If they are *not* isometries, provide a counter-example, and if they are, provide a proof.

(a) $f(x,y) = (-2x, x+y)$,

(b) $f(x, y) = (\cos(x), y)$,

(c) $f(x, y) = (x^2, y)$,

(d) $f(x, y) = (y, x)$,

(e) $f(x, y) = (-x, -y)$,

(f) $f(x, y) = (x, xy)$,

(a) Not an ismoetry. Consider $P = (0, 0), Q = (1, 0)$. Then $d(f(P), f(Q)) = d((0, 0), (-2, 1)) = \sqrt{4 + 1} \neq 1 = d(P, Q)$.

(b) Not an isometry. Consider $P = (0, 0), Q = (2\pi, 0)$. Then $d(f(P), f(Q)) = d((1, 0), (1, 0)) = 0 \neq 2\pi = d(P, Q)$.

(c) Not an isometry. Consider $P = (0, 0), Q = (2, 0)$. Then $d(f(P), f(Q)) = d((0, 0), (4, 0)) = 4 \neq 2 = d(P, Q)$.

(d) This is an isometry. $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} = d((y_1, x_1), (y_2, x_2)) = d(f(x_1, y_1), f(x_2, y_2))$.

(e) This is an isometry. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-x_2 - (-x_1))^2 + (-y_2 - (-y_1))^2} = d((-x_1, -y_1), (-x_2, -y_2)) = d(f(x_1, y_1), f(x_2, y_2))$.

(f) This is not an isometry. Consider $P = (0, 0), Q = (0, 1)$. $d(f(P), f(Q)) = d((0, 0), (0, 0)) = 0 \neq 1 = d(P, Q)$.

Problem 3. (25 pts) Let $P = (3, 4) \in \mathbb{R}^2$ be a point and $L \subseteq \mathbb{R}^2$ be the line

$$L = \{(x, y) : y = \sqrt{3}x - \sqrt{3} + 2\}.$$

(a) Let $R_{\pi/3,P}$ be the counter-clockwise rotation by $\pi/3$-radians centered at P. Find a formula for the isometry $R_{\pi/3,P}$.

(b) Where does the point $(-2, -7)$ map under $R_{\pi/3,P}$?

(c) Let $\overline{r}_L$ be the reflection along the line L. Find a formula for the isometry $\overline{r}_L$.

(d) Describe where the points $(1, 2)$, $(-2, -7)$ and $(3, 4)$ map under the isometry $\overline{r}_L$.

(e) Consider the composition $R_{\pi/3,P} \circ \overline{r}_L$. Where does the origin $(0, 0) \in \mathbb{R}^2$ map to ?

(f) Consider the composition $\overline{r}_L \circ R_{\pi/3,P}$. Compute the image of the origin $(0, 0)$ under this isometry and compare with Part (e).

(a) $R_{\pi/3,P} = t_{(3,4)} R_{\pi/3,O} t_{(-3,-4)} = t_{(3,4)} \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} t_{(-3,-4)} = t_{(3,4)} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} t_{(-3,-4)}$, so

$$\begin{aligned} R_{\pi/3,P} \begin{bmatrix} x \\ y \end{bmatrix} &= t_{(3,4)} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} t_{(-3,-4)} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= t_{(3,4)} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x-3 \\ y-4 \end{bmatrix} \\ &= t_{(3,4)} \begin{bmatrix} \frac{x-3-\sqrt{3}(y-4)}{2} \\ \frac{\sqrt{3}(x-3)+y-4}{2} \end{bmatrix} = \begin{bmatrix} \frac{x+3-\sqrt{3}(y-4)}{2} \\ \frac{\sqrt{3}(x-3)+y+4}{2} \end{bmatrix} \end{aligned}$$

(b) $R_{\pi/3,P}(-2,-7) = \begin{bmatrix} \frac{-2+3-\sqrt{3}(-7-4)}{2} \\ \frac{\sqrt{3}(-2-3)-7+4}{2} \end{bmatrix} = \begin{bmatrix} \frac{1+11\sqrt{3}}{2} \\ \frac{-5\sqrt{3}-3}{2} \end{bmatrix}$

(c) The line L passes through $(0, -\sqrt{3}+2), (1,2)$, so we can compute that the angle between L and the x-axis is $\theta = \arctan(\sqrt{3}) = \pi/3$. We apply the isometry $R_{-\pi/3} \circ t_{(-1,-2)}$ to shift L to the x-axis. Thus,

$$\begin{aligned} \overline{r}_L(x,y) &= t_{(1,2)} \circ R_{\pi/3} \circ \overline{r} \circ R_{-\pi/3} \circ t_{(-1,-2)}(x,y) \\ &= t_{(1,2)} \circ \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} \circ \overline{r} \circ \begin{bmatrix} \cos -\pi/3 & -\sin -\pi/3 \\ \sin -\pi/3 & \cos -\pi/3 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= t_{(1,2)} \circ \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \circ \overline{r} \circ \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= t_{(1,2)} \circ \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \circ \overline{r} \begin{bmatrix} \frac{x-1+\sqrt{3}y-2\sqrt{3}}{2} \\ \frac{-\sqrt{3}x+\sqrt{3}+y-2}{2} \end{bmatrix} \\ &= t_{(1,2)} \circ \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} \frac{x-1+\sqrt{3}y-2\sqrt{3}}{2} \\ \frac{\sqrt{3}x-\sqrt{3}-y+2}{2} \end{bmatrix} \\ &= t_{(1,2)} \begin{bmatrix} \frac{x-1+\sqrt{3}y-2\sqrt{3}-3x+3+\sqrt{3}y-2\sqrt{3}}{4} \\ \frac{\sqrt{3}x-\sqrt{3}+3y-6+\sqrt{3}x-\sqrt{3}-y+2}{4} \end{bmatrix} \\ &= t_{(1,2)} \begin{bmatrix} \frac{-2x+2+2\sqrt{3}y-4\sqrt{3}}{4} \\ \frac{2\sqrt{3}x+2y-2\sqrt{3}-4}{4} \end{bmatrix} = t_{(1,2)} \begin{bmatrix} \frac{-x+1+\sqrt{3}y-2\sqrt{3}}{2} \\ \frac{\sqrt{3}x+y-\sqrt{3}-2}{2} \end{bmatrix} = \begin{bmatrix} \frac{-x+3+\sqrt{3}y-2\sqrt{3}}{2} \\ \frac{\sqrt{3}x+y-\sqrt{3}+2}{2} \end{bmatrix} \end{aligned}$$

(d)

$$\begin{aligned} \overline{r}_L(1,2) &= (1,2), \text{ since } (1,2) \in L. \\ \overline{r}_L(-2,-7) &= \begin{bmatrix} \frac{2+3-7\sqrt{3}-2\sqrt{3}}{2} \\ \frac{-2\sqrt{3}-7-\sqrt{3}+2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-9\sqrt{3}}{2} \\ \frac{-3\sqrt{3}-5}{2} \end{bmatrix} \\ \overline{r}_L(3,4) &= \begin{bmatrix} \frac{-3+3+4\sqrt{3}-2\sqrt{3}}{2} \\ \frac{3\sqrt{3}+4-\sqrt{3}+2}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ \sqrt{3}+3 \end{bmatrix} \end{aligned}$$

(e) We compute:

$$\begin{aligned}
R_{\pi/3,P} \circ \overline{r}_L(0,0) &= R_{\pi/3,P} \begin{bmatrix} \frac{3-2\sqrt{3}}{2} \\ \frac{-\sqrt{3}+2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3-2\sqrt{3})/2+3-\sqrt{3}(-\sqrt{3}+2)/2+4\sqrt{3}}{2} \\ \frac{\sqrt{3}(3-2\sqrt{3})/2-3\sqrt{3}+(-\sqrt{3}+2)/2+4}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\frac{3}{2}-\sqrt{3}+3+\frac{3}{2}-\sqrt{3}+4\sqrt{3}}{2} \\ \frac{\frac{3\sqrt{3}}{2}-3-3\sqrt{3}-\frac{\sqrt{3}}{2}+5}{2} \end{bmatrix} = \begin{bmatrix} 3+\sqrt{3} \\ 1-\sqrt{3} \end{bmatrix}
\end{aligned}$$

(f) We compute:

$$\begin{aligned}
\overline{r}_L \circ R_{\pi/3,P}(0,0) &= \overline{r}_L \begin{bmatrix} \frac{3+4\sqrt{3}}{2} \\ \frac{-3\sqrt{3}+4}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-(3+4\sqrt{3})/2+3+\sqrt{3}(-3\sqrt{3}+4)/2-2\sqrt{3}}{2} \\ \frac{\sqrt{3}(3+4\sqrt{3})/2+(-3\sqrt{3}+4)/2-\sqrt{3}+2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\frac{3}{2}-2\sqrt{3}+3-\frac{9}{2}+2\sqrt{3}-2\sqrt{3}}{2} \\ \frac{\frac{3\sqrt{3}}{2}+6-\frac{3\sqrt{3}}{2}+2-\sqrt{3}+2}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{2}-\sqrt{3} \\ 5-\frac{\sqrt{3}}{2} \end{bmatrix}
\end{aligned}$$

We see that we get a different point than we did in part (e), hence $\overline{r}_L$ and $R_{\pi/3,P}$ do not commute.

Problem 4. (25 pts) In this problem we explore basic compositions of rotations and translations. Solve the following parts:

(a) Let $\theta, \phi \in S^1$ be two angles. Show that

$$R_\theta \circ R_\phi = R_{\theta+\phi}.$$

(b) Let $\theta \in S^1$ be an angle. Find the unique angle $\phi \in S^1$ such that $R_\theta \circ R_\phi = \mathrm{Id}$ is the identity map $\mathrm{Id}(x,y) = (x,y)$.

(c) Let (α,β) and (γ,δ) be two points in the Euclidean Plane $\mathbb{R}^2$. Prove that

$$t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)}.$$

(d) Let $(\alpha,\beta) \in \mathbb{R}^2$ be a point in Euclidean plane. Find the unique $(\gamma,\delta) \in \mathbb{R}^2$ such that the composition $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = \mathrm{Id}$.

(a) Recall we have the angle sum formulas: $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$, $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. Then, we observe that

$$\begin{aligned} R_\theta \circ R_\phi &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta(-\sin\phi) - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = R_{\theta+\phi}. \end{aligned}$$

(b) Note that $S^1 = \{[0, 2\pi] : 0 \sim 2\pi\}$. If $\theta = 0$, then $\phi = 0$ is the unique point so that $R_\theta \circ R_\phi = R_0 = \text{Id}$. Indeed, if $\phi \in S^1 \setminus \{0\}$, then $R_\theta \circ R_\phi$ is a nontrivial rotation, hence not the identity. Now assume $\theta \in S^1 \setminus \{0\}$. Then in order for $R_\theta \circ R_\phi$ to be the identity, we must have $\theta + \phi \equiv 0 \pmod{2\pi}$, so $\phi = -\theta + n \cdot 2\pi$ for some $n \in \mathbb{Z}$. But since $\theta \in (0, 2\pi)$, the only n satisfying $\phi = -\theta + n \cdot 2\pi \in S^1$ is $n = 1$, so $\phi = 2\pi - \theta$ is the unique angle such that $R_\theta \circ R_\phi = \text{Id}$.

(c) $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)}(x, y) = t_{(\alpha,\beta)}(x + \gamma, y + \delta) = (x + \alpha + \gamma, y + \beta + \delta) = t_{(\alpha+\gamma,\beta+\delta)}(x, y)$.

(d) We have $\text{Id} = t_{(0,0)} = t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)} \iff \alpha + \gamma = 0, \beta + \delta = 0 \iff (\gamma, \delta) = (-\alpha, -\beta)$, so this is the unique point such that $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = \text{Id}$.

Problem 5. (ungraded) Let $L = \{(x, y) : y = 0\} \subseteq \mathbb{R}^2$ and $M = \{(x, y) : x = 0\} \subseteq \mathbb{R}^2$ be the x and y-axis respectively.

(a) Show that $\overline{r}_L \overline{r}_M(x, y) = (-x, -y)$.

(b) Prove that there exists no line $N \subseteq \mathbb{R}^2$ such that

$$\overline{r}_N = \overline{r}_L \overline{r}_M,$$

where L, M are as in Part (a). Thus, we learn that the composition of reflections is *not* always a reflection.

(c) Find an angle $\phi \in S^1$ such that the composition $\overline{r}_L \overline{r}_M$ in Part (a) equals the rotation R_ϕ, i.e. $R_\phi = \overline{r}_L \overline{r}_M$. Thus, we learn that the composition of reflections can *sometimes* be a rotation.

(d) Find all the angles $\theta \in S^1$, if any, such that the rotation R_θ, centered at the origin, *commutes* with any reflection $\overline{r}_L$, where L is a line through the origin:

$$R_\theta \circ \overline{r}_L = \overline{r}_L \circ R_\theta.$$

(a)

$$\begin{aligned}
\overline{r}_L\overline{r}_M(x,y) &= R_{-\pi/2,O} \circ \overline{r}_M \circ R_{\pi/2,O} \circ \overline{r}_M(x,y) \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \circ \overline{r}_M \circ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \circ \overline{r}_M \begin{bmatrix} -y \\ -x \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -y \\ x \end{bmatrix} = (-x,-y)
\end{aligned}$$

(b) Note that $\overline{r}_N$ fixes infinitely many points (i.e. the points on the line N). But if (x,y) is a fixed point of $\overline{r}_L\overline{r}_M$, then $(x,y) = \overline{r}_L\overline{r}_M(x,y) = (-x,-y) \iff (x,y) = (0,0)$, so $\overline{r}_L\overline{r}_M$ only has one fixed point. Therefore, there exists no line N such that $\overline{r}_N = \overline{r}_L\overline{r}_M$.

(c) Take $\phi = \pi$. Then $R_\phi(x,y) = \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (-x,-y)$, thus $R_\phi = \overline{r}_L\overline{r}_M$.

(d) By conjugation, we only need to consider the case where L is the x-axis. For details regarding the conjugation, we observe that if L is not the x-axis, then we may write $\overline{r}_L = R_\phi \circ \overline{r} \circ R_{-\phi}$ for some $\phi \in S^1$. Then,

$$\begin{aligned}
& R_\theta \circ \overline{r}_L = \overline{r}_L \circ R_\theta \\
\iff & R_\theta \circ R_\phi \circ \overline{r} \circ R_{-\phi} = R_\phi \circ \overline{r} \circ R_{-\phi} \circ R_\theta \\
\iff & R_{\theta+\phi} \circ \overline{r} \circ R_{-\phi} = R_\phi \circ \overline{r} \circ R_{\theta-\phi} \\
\iff & R_{-\phi} \circ R_{\theta+\phi} \circ \overline{r} \circ R_{-\phi} \circ R_\phi = R_{-\phi} \circ R_\phi \circ \overline{r} \circ R_{\theta-\phi} \circ R_\phi \\
\iff & R_\theta \circ \overline{r} = \overline{r} \circ R_\theta.
\end{aligned}$$

To find θ such that $R_\theta \circ \overline{r} = \overline{r} \circ R_\theta$, it suffices to find θ such that $\overline{r}R_\theta\overline{r} = R_\theta$ (by composing on the left by $\overline{r}$). We observe:

$$\begin{aligned}
\overline{r}R_\theta\overline{r}(x,y) &= \overline{r} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} \\
&= \overline{r} \begin{bmatrix} x\cos\theta - y\sin\theta \\ -x\sin\theta - y\cos\theta \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \\
&= R_\theta(x,y) = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix} \\
\implies & -y\sin\theta = y\sin\theta, -x\sin\theta = x\sin\theta \\
\implies & 2\sin\theta = 0 \implies \theta = 0, \pi.
\end{aligned}$$

So the angles in S^1 such that the rotation R_θ commutes with any reflection $\overline{r}_L$, where L is a line through the origin, are $0, \pi$.

Problem 6. (ungraded) Consider the square $S \subseteq \mathbb{R}^2$ with four vertices given by the points $(-1,-1), (-1,1), (1,1), (1,-1) \in \mathbb{R}^2$. The square consists of the convex hull of these four points, i.e. the region given by

$$S = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

(a) Show that there exists *no* translation $t_{(\alpha,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, except for the identity, such that $t_{(\alpha,\beta)}(S) \subseteq S$, i.e. the translation sends the square to the square.

(b) Find *four* distinct lines $L_1, L_2, L_3, L_4 \subseteq \mathbb{R}^2$ such that the reflections $\overline{r}_{L_i}$, $1 \leq i \leq 4$, all satisfy the inclusion $\overline{r}_{L_i}(S) \subseteq S$.

(c) Find 27 distinct lines $M \subseteq \mathbb{R}^2$ such that $\overline{r}_M(S) \nsubseteq S$, i.e. the reflection $\overline{r}_M$ maps the square S *not inside* the square S.

(d) Find all angles $\theta \in S^1$ such that $R_\theta(S) \subseteq S$.

(e) Find *infinitely many* angles $\theta \in S^1$ such that $R_\theta(S) \nsubseteq S$.

(a) We consider the diagonal $y = x$ for $-1 \leq x, y \leq 1$, whose endpoints are $(-1,-1),(1,1)$. If $\alpha > 0$ or $\beta > 0$, then clearly $t_{(\alpha,\beta)}(1,1) \notin S$. If $\alpha < 0$ or $\beta < 0$, then likewise $t_{(\alpha,\beta)}(-1,-1) \notin S$, so we must have $\alpha = \beta = 0$.

(b) The lines $L_1 = \{y = 0\}, L_2 = \{x = 0\}, L_3 = \{y = x\}, L_4 = \{y = -x\}$ are four distinct lines such that $\overline{r}_{L_i}(S) \subseteq S$.

(c) The question doesn't require that M must pass through the origin (though one can certainly find many such lines through the origin), so we can just take $M_i := \{y = 200 + i\}$ for $1 \leq i \leq 27$ or something crazy like this.

(d) Such a rotation must send vertices to vertices, so $(1,1)$ must be mapped to $(1,1), (-1,1), (-1,-1)$, or $(1,-1)$, and these would correspond to rotations by $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ respectively.

(e) From part (d), we've already seen all the angles so that containment holds. Thus, for any $\theta \in S^1 \setminus \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, we have $R_\theta(S) \nsubseteq S$.

Problem 7. (25 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) The linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is an isometry.

(b) Any linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with $a \neq 0$, must be an isometry.

(c) The composition $f \circ g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of two isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is always an isometry.

(d) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry. If there exist infinitely many points $P \in \mathbb{R}^2$ such $f(P) = P$, then $f = \mathrm{Id}$ must be the identity.

(e) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isometry which fixes the points $(0,0)$, $(1,0)$ and $(0,1)$, i.e. $f(0,0) = (0,0)$, $f(1,0) = (1,0)$ and $f(0,1) = (0,1)$. Then $f = \mathrm{Id}$ must be the identity.

(f) The composition of reflections is *always* a reflection.

(g) The composition of rotations centered at the origin are *always* rotations.

(h) The composition of translations is *always* a translation.

(i) There is an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that sends the square S, as defined in Problem 6, strictly inside itself, i.e.

$$f(S) \subseteq \{(x,y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\}.$$

(j) For any rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, there exists a power $n \in \mathbb{N}$ such that the composition $R_\theta^n = \mathrm{Id}$.

(a) True. f is the rotation by $\frac{\pi}{2}$, as seen in the formula $R_{\frac{\pi}{2}} = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is an isometry.

(b) False. Let $a = 1$, and consider $P = (0,0), Q = (0,1)$. Then $d(f(P), f(Q)) = d((0,0),(1,1)) = \sqrt{2} \neq 1 = d(P,Q)$, hence f is not an isometry.

(c) True. If f, g are both isometries, then $d(f(g(P)), f(g(Q))) = d(g(P), g(Q)) = d(P,Q)$, hence $f \circ g$ is also an isometry.

(d) False. Any reflection fixes infinitely many points (the points on the line of reflection), but does not fix any other point, thus is not the identity.

(e) True. By linearity, $f(x,y) = f(x,0) + f(0,y) = xf(1,0) + yf(0,1) = x(1,0) + y(0,1) = (x,y)$, so f fixes all points, and is therefore the identity.

(f) False. Any reflection composed with itself is the identity, which is not a reflection.

(g) True. This is shown in 4a.

(h) True. This is shown in 4c.

(i) False. We use $\mathring{S}$ to denote the interior of S. Assume for a contradiction that such an f exists. First observe that the distance between $(-1,-1),(1,1) \in S$ is $\sqrt{8}$. So f must map these to two points in $\mathring{S}$, while retaining a distance of $\sqrt{8}$. We show that any two points $(x_1, y_1),(x_2, y_2) \in \mathring{S}$ must have distance strictly less than $\sqrt{8}$. Indeed, we have $-1 < x_1, x_2, y_1, y_2 < 1$, and so $|x_2 - x_1| < 1-(-1) = 2, |y_2 - y_1| < 1-(-1) = 2$. In particular, $(x_2 - x_1)^2 < 4$ and $(y_2 - y_1)^2 < 4$. It follows that $d((x_1, y_1),(x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} < \sqrt{4+4} = \sqrt{8}$. Therefore, such an f does not exist.

(j) False. Consider R_θ for $\theta = \sqrt{2}\pi$. If such an n exists so that $\mathrm{Id} = R_\theta^n = R_{n\theta}$, then we must have $n\theta \equiv 0 \pmod{2\pi}$. That is, $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$. But since $\theta = \sqrt{2}\pi$, we have $n\sqrt{2}\pi = 2k\pi \implies \sqrt{2} = \frac{2k}{n} \in \mathbb{Q}$, which is a contradiction. (note that we were allowed to divide by n because $n \in \mathbb{N}$, so $n \neq 0$)