MAT 141: PROBLEM SET 3

DUE TO FRIDAY APR 24 AT 10:00AM

Abstract. This is the third problem set for the Euclidean and Non-Euclidean Geometry Course in the Spring Quarter 2026. It is due Friday Apr 24 at 11:00am via online submission.

Purpose: The goal of this assignment is to practice problems on the geometry of Euclidean Surfaces $\mathbb{R}^2/\Gamma$. In particular, we would like to become familiar with the geometry of the cylinder, the *twisted* cylinder, the 2-torus and the Klein bottle.

Task: Solve Problems 1 through 7 below. Problems 1,2 and 6 will not be graded but I trust that you will work on them. Problems 3, 4, 5 and 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $\Gamma \subseteq \text{Iso}(R^2)$ be a subgroup of the isometry group of the Euclidean Plane $\mathbb{R}^2$. Consider the set of Γ-orbits $\mathbb{R}^2/\Gamma$ and define the distance $\mathbb{R}^2/\Gamma$ as follows:

$$d(\Gamma(P), \Gamma(Q)) := \min_{P', Q' \in \mathbb{R}^2} \{d(P', Q') : P' \in \Gamma(P), Q' \in \Gamma(Q)\},$$

where $P, Q \in \mathbb{R}^2$. Show that we have the equality

$$d(\Gamma(P), \Gamma(Q)) = \min_{Q' \in \mathbb{R}^2} \{d(P, Q') : Q' \in \Gamma(Q)\}.$$

Problem 2. Let $S \subseteq \text{Iso}(\mathbb{R}^2)$ be a subset of the isometry group, and let $\langle S \rangle \subseteq \text{Iso}(\mathbb{R}^2)$ denote the group generated by elements in S.

(a) If $S = \{t_{(1,0)}\}$ show that

$$\langle t_{(1,0)} \rangle = \{t_{(n,0)} : n \in \mathbb{Z}\}.$$

(b) If $S = \{t_{(1,0)}, t_{(0,1)}\}$ show that
$$\langle S \rangle = \{t_{(n,m)} : (n,m) \in \mathbb{Z}^2\}.$$
(c) If $S = \{t_{(1,0)}, t_{(0,1)}\}$, $\Gamma = \langle S \rangle$, and $P = (0,0)$ show that
$$\Gamma(P) = \{(n,m) \in \mathbb{Z}^2\}.$$
(d) If $S = \{t_{(1,0)} \circ \overline{r}\}$, $\Gamma = \langle S \rangle$, and $P = (0,0)$ show that
$$\Gamma(P) = \{(n,0) \in \mathbb{Z}^2\}.$$
(e) If $S = \{t_{(1,0)} \circ \overline{r}\}$, $\Gamma = \langle S \rangle$, and $P = (0,1)$ show that
$$\Gamma(P) = \{(n,(-1)^n) \in \mathbb{Z}^2\}.$$

Problem 3. (20 pts) (**The Cylinder**) Let $C = \mathbb{R}^2/\Gamma$ be the Euclidean cylinder, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the translation $t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. We shall use coordinates $(x,y) \in C$, induced by the coordinates $(x,y) \in \mathbb{R}^2$, where in the cylinder C we identify $(x,y) \sim (x+n,y)$ for any $n \in \mathbb{Z}$.

Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the cylinder C is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) For each of the following four pairs of points $P, Q \in C$, compute the distance $d_C(P,Q)$ between these points in the cylinder.
$$P = (0.5, 0.5), Q = (0.5, 0.7), \qquad P = (0,0), Q = (0,0.3)$$
$$P = (0.1, 0.2), Q = (0.9, 0.3), \qquad P = (0.1, 0.1), Q = (0.9, 0.8).$$
(b) What is the distance between $(1/3, 1/2)$ and $(7/3, -11/2)$? And the distance between $(0,0)$ and $(3,4)$?

(c) Give example of three points $P \in \mathbb{R}^2$ such that $\pi(P) = (0, 0.5)$.

(d) Find infinitely points $P \in \mathbb{R}^2$ such that $\pi(P) = (0.2, 4) \in C$.

(e) (Exercise 2.2.1) Which of the following properties of euclidean lines hold for lines on the cylinder ?

(i) There is a line through any two points.

(ii) There is a unique line through any two points.

(iii) Two lines meet in at most one point.

(iv) There are lines which do not meet.

(v) A line has infinite length.

(vi) A line gives the shortest distance between two points.

(vii) A line does not cross itself.

Problem 4. (20 pts) (**The Möbius Band**) Let $M = \mathbb{R}^2/\Gamma$ be the Euclidean twisted cylinder, where $\Gamma = \langle t_{(1,0)} \circ \overline{r} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the glide reflection $t_{(1,0)} \circ \overline{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. We shall use coordinates $(x, y) \in M$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the twisted cylinder M we identify $(x, y) \sim (x + n, (-1)^n y)$, for any value $n \in \mathbb{Z}$.

Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the twisted cylinder M is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) Let $P = (0.1, 0), Q = (0.9, 0.1), R = (0.1, 0.9), S = (-0.1, -0.9), T = (0.9, 0.9)$ be points in the twisted cylinder M. Find their pairwise distances. (There are ten of them.)

(b) Consider the lines $L = \{x = 0.5\} \subseteq \mathbb{R}^2$ and $K = \{x = y\} \subseteq \mathbb{R}^2$. Find the intersection points of $\pi(L)$ and $\pi(K)$.

(c) Let $N = \{(x, y) \in \mathbb{R}^2 : x = -y\} \subseteq \mathbb{R}^2$. Find the number of intersection points of $\pi(K)$ and $\pi(N)$.

(d) Find two lines in the twisted cylinder M which do *not* intersect.

(e) Is it possible for two lines L_1, L_2 in the twisted cylinder M to have exactly one intersection point, i.e. $|L_1 \cap L_2| = 1$?

(f) Show that there is no continuous bijection $g : C \longrightarrow M$ between the cylinder C and the twisted cylinder M such that g sends the d_C-distance between any two points $P, Q \in C$, equals the d_M-distance between $g(P), g(Q) \in M$.

Problem 5. (20 pts) (**The Torus**) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

We use coordinates $(x, y) \in T^2$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the torus T^2 we identify $(x, y) \sim (x + n, y + m)$ for any $(n, m) \in \mathbb{Z}^2$. Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the torus T^2 is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) Find the distance between the two points $(1.1, -12), (-1.9, 23.8) \in T^2$.

(b) Let $L_\alpha = \{x = \alpha y\} \subseteq \mathbb{R}^2$ be the line of slope α. Suppose α is a non-zero rational number. Show that the intersection of $\pi(L_\alpha)$ and $\pi(L_0)$ consists of finitely many points.

(c) Assume α is a non-zero irrational number. How many times do the lines $\pi(L_\alpha)$ and $\pi(L_0)$ intersect ?

(d) Find four lines $L_1, L_2, L_3, L_4 \subseteq T^2$ such that $|L_i \cap L_j| = \emptyset$ for $1 \leq i, j \leq 4$, $i \neq j$, i.e. four lines which they do *not* pairwise intersect.

(e) For any $\alpha \in \mathbb{R}$, find infinitely many lines $L \subseteq \mathbb{R}^2$ such that $\pi(L) = L_\alpha$.

Problem 6. (**The Klein Bottle**) Let $K = \mathbb{R}^2/\Gamma$ be the Klein bottle, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \circ \overline{r} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by a translation and a glide reflection. We use coordinates $(x, y) \in K$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the Klein bottle K we identify $(x, y) \sim (x+n, (-1)^n y)$ and also $(x, y) \sim (x, y+n)$, for any $n \in \mathbb{Z}$. Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the Klein Bottle is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) Draw the Γ-orbits $\Gamma(P), \Gamma(Q), \Gamma(R) \subseteq \mathbb{R}^2$ for each of the three points $P = (0, 0)$ and $Q = (1, 0)$ and $R = (-1, 0)$ in the Euclidean plane $\mathbb{R}^2$.

(b) Find the distance between the two points $(0, 0), (0.9, 0.9) \in K$.

(c) Show that there exist parallel lines inside a Klein Bottle.

(d) Consider the lines $L = \{y = 0\} \subseteq \mathbb{R}^2$ and $W = \{x = y\} \subseteq \mathbb{R}^2$. Find the intersection points of $\pi(L)$ and $\pi(W)$.

(e) Let $N = \{(x, y) \in \mathbb{R}^2 : y = 0.5\} \subseteq \mathbb{R}^2$. Find the number of intersection points of $\pi(W)$ and $\pi(N)$.

(f) Show that the Klein bottle can be cut into two Möbius bands.

Problem 7. (25 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) For each line $\mathcal{L} \subseteq C$ in the cylinder, there are infinitely many lines $L \subseteq \mathbb{R}^2$ such that $\pi(L) = \mathcal{L}$.

(b) For each point $p \subseteq C$ in the cylinder, there are infinitely many points $P \subseteq \mathbb{R}^2$ such that $\pi(P) = p$.

(c) For each line $\mathcal{L} \subseteq T^2$ in the 2-torus, there are infinitely many lines $L \subseteq \mathbb{R}^2$ such that $\pi(L) = \mathcal{L}$.

(d) The only isometry of the twisted cylinder M is the identity.

(e) Any isometry of the cylinder C must be a vertical translation, i.e. an integer multiple of the isometry $t_{(0,1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, descended to $t_{(0,1)} : \mathbb{R}^2/\Gamma \longrightarrow \mathbb{R}^2/\Gamma$.

(f) An isometry $f : T^2 \longrightarrow T^2$ is uniquely determined by the three images

$$f(A), f(B), f(C) \in T^2$$

of three non-collinear points $A, B, C \in T^2$.

(g) Let $P, Q \in C$ be two points in the cylinder. Then there exists an isometry $f : C \longrightarrow C$ such that $f(P) = Q$.

(h) Let $P, Q \in T^2$ be two points in the torus. Then there exists an isometry $f : T^2 \longrightarrow T^2$ such that $f(P) = Q$.

(i) Let $P, Q \in C$ be distinct, then the set of points $R \in C$ which are equidistant to P and Q form a line in C, i.e. $\{R \in C : d(R, P) = d(R, Q)\}$ is a line.

(j) Let $P, Q \in T^2$ be distinct, then the set of points $R \in T^2$ which are equidistant to P and Q form a line in T^2, i.e. $\{R \in T^2 : d(R, P) = d(R, Q)\}$ is a line.