MAT 141: PROBLEM SET 4

DUE TO FRIDAY MAY 15 AT 11:00AM

Abstract. This is the fourth problem set for the Euclidean and Non-Euclidean Geometry Course in the Spring Quarter 2026. It is due Friday May 15 at 11:00am via online submission.

1. Instructions

Purpose: The goal of this assignment is to practice problems on the geometry of the Hyperbolic Plane $\mathbb{H}^2$. In particular, we would like to become familiar with hyperbolic lengths, hyperbolic isometries and hyperbolic lines.

Task: Solve Problems 1 through 5 below. Problem 2 will not be graded but I trust that you will work on it. Problems 1,3,4 and 5 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Mathematical commments: Every instance of "length" or "distance" refers to *hyperbolic lengths* and *hyperbolic distance*. Similarly, all isometries are taken to be hyperbolic isometries. We use the notation $z \in \mathbb{H}^2$ to indicate complex coordinates in the hyperbolic upper-half plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

2. Problems

Problem 1. (25 points) Let $P = 3 + 4i$ and $Q = -3 + 4i$ be two points in the hyperbolic upper-half plane $\mathbb{H}^2$. Consider the following two paths from P to Q:

$$\gamma_1 : [0, 1] \longrightarrow \mathbb{H}^2, \quad \gamma_1(t) = (3 - 6t) + 4i,$$

$$\gamma_2 : [\theta_1, \theta_2] \longrightarrow \mathbb{H}^2, \quad \gamma_1(t) = 5\cos(t) + 5i \sin(t),$$

where $\theta_1 = \arccos(0.6)$ and $\theta_2 = \arccos(-0.6)$.

(a) Draw the image of γ_1, γ_2 in the hyperbolic plane $\mathbb{H}^2$.

(b) Compute the hyperbolic lengths of γ_1 and γ_2.

(c) Show that the hyperbolic distance between P and Q is less equal than $\ln(4)$.

Problem 2. Let $a, b, c, d \in \mathbb{R}$ be given such that $ad - bc = 1$. Consider the map

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \quad f(z) = \frac{az + b}{cz + d}.$$

(a) Show that the restriction of f to $\mathbb{H}^2$ yields a map

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2.$$

That is, $\operatorname{Im}(f(z)) > 0$ if $\operatorname{Im}(z) > 0$.

(b) Show that f is a hyperbolic isometry.

Problem 3. (25 points) Consider the hyperbolic isometry

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{1}{\bar{z}}.$$

(a) Compute the image of the points of the form ki and $\frac{i}{k}$ where $k \in \mathbb{N}$.

(b) Show that all the points of the unit half-circle

$$S := \{z \in \mathbb{H}^2 : |z| = 1\}$$

are fixed under f.

(c) Prove that S exactly coincides with the set of fixed points of f.

Problem 4. (25 points) Consider the family of hyperbolic isometries

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{az + b}{cz + d}, \tag{2.1}$$

where $a, b, c, d \in \mathbb{R}$ be given such that $ad - bc = 1$.

(a) Let $P, Q \in \mathbb{H}^2$ be two points, show that there exists a hyperbolic isometry of the form (2.1) above such that $f(P) = Q$.

(b) Let $P \in \mathbb{H}^2$ be an arbitrary point. Describe all the isometries of the form (2.1) such that $f(P) = P$.

Problem 5. (25 points) Let $a, b, c, d \in \mathbb{R}$ be given such that $ad - bc = 1$. Consider the maps

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{-1}{z}.$$

$$g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad g(z) = \frac{-1}{z+1}.$$

(a) Find the fixed points of f and g.

(b) Describe f and g geometrically as well as you can.

(c) Show that $f^2 := f \circ f$ is the identity.

(d) Show that $g^3 := g \circ g \circ g$ is the identity.

(g) Consider the composition $h = f \circ g$. Describe h geometrically and rigorously show that there exists no $n \in \mathbb{N}$ such that h^n is the identity.