

MAT 141: PROBLEM SET 4

DUE TO FRIDAY MAY 15 AT 11:00AM

ABSTRACT. This is the fourth problem set for the Euclidean and Non-Euclidean Geometry Course in the Spring Quarter 2026. It is due Friday May 15 at 11:00am via online submission.

1. INSTRUCTIONS

Purpose: The goal of this assignment is to practice problems on the geometry of the Hyperbolic Plane \mathbb{H}^2 . In particular, we would like to become familiar with hyperbolic lengths, hyperbolic isometries and hyperbolic lines.

Task: Solve Problems 1 through 5 below. Problem 2 will not be graded but I trust that you will work on it. Problems 1,3,4 and 5 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use “Geometry of Surfaces” by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Mathematical comments: Every instance of “length” or “distance” refers to *hyperbolic lengths* and *hyperbolic distance*. Similarly, all isometries are taken to be hyperbolic isometries. We use the notation $z \in \mathbb{H}^2$ to indicate complex coordinates in the hyperbolic upper-half plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

2. PROBLEMS

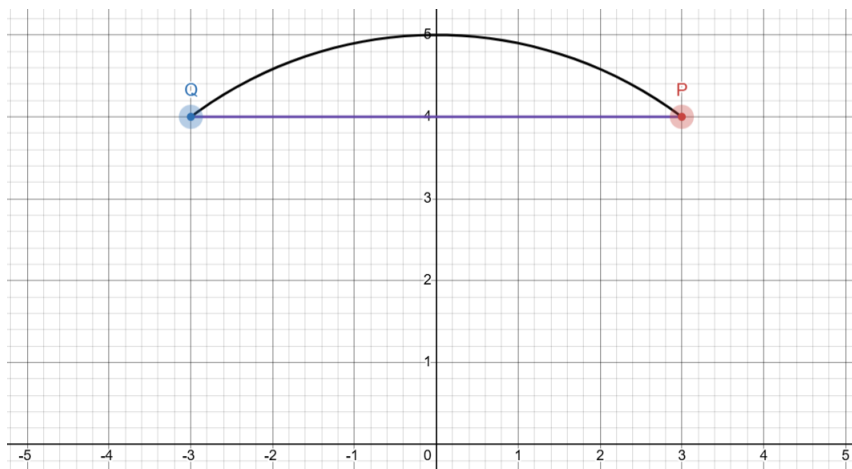
Problem 1. (25 points) Let $P = 3 + 4i$ and $Q = -3 + 4i$ be two points in the hyperbolic upper-half plane \mathbb{H}^2 . Consider the following two paths from P to Q :

$$\gamma_1 : [0, 1] \longrightarrow \mathbb{H}^2, \quad \gamma_1(t) = (3 - 6t) + 4i,$$

$$\gamma_2 : [\theta_1, \theta_2] \longrightarrow \mathbb{H}^2, \quad \gamma_2(t) = 5 \cos(t) + 5i \sin(t),$$

where $\theta_1 = \arccos(0.6)$ and $\theta_2 = \arccos(-0.6)$.

- Draw the image of γ_1, γ_2 in the hyperbolic plane \mathbb{H}^2 .
- Compute the hyperbolic lengths of γ_1 and γ_2 .
- Show that the hyperbolic distance between P and Q is less equal than $\ln(4)$.



- The purple path is γ_1 and the black path is γ_2 .
- We use the formula

$$ds = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y} dt.$$

For γ_1 , we have $x(t) = 3 - 6t, y(t) = 4$, thus $x'(t) = -6, y'(t) = 0$. Then the length of γ_1 is

$$\int_0^1 \frac{\sqrt{(-6)^2 + 0}}{4} dt = \int_0^1 \frac{6}{4} dt = \frac{3}{2}.$$

For γ_2 , we have $x(t) = 5 \cos(t), y(t) = 5 \sin(t)$, so $x'(t) = -5 \sin(t), y'(t) = 5 \cos(t)$. Then the length of γ_2 is

$$\begin{aligned} \int_{\arccos(0.6)}^{\arccos(-0.6)} \frac{\sqrt{25 \cos^2(t) + 25 \sin^2(t)}}{5 \sin(t)} dt &= \int_{\arccos(0.6)}^{\arccos(-0.6)} \frac{\sqrt{25}}{5 \sin(t)} dt \\ &= \int_{\arccos(0.6)}^{\arccos(-0.6)} \frac{1}{\sin(t)} dt \\ &= \ln \left(\tan \left(\frac{t}{2} \right) \right) \Big|_{\arccos(0.6)}^{\arccos(-0.6)} \\ &= \ln \left(\frac{\sin(t)}{1 + \cos(t)} \right) \Big|_{\arccos(0.6)}^{\arccos(-0.6)} \\ &= \ln \left(\frac{\frac{4}{5}}{1 - \frac{3}{5}} \right) - \ln \left(\frac{\frac{4}{5}}{1 + \frac{3}{5}} \right) \\ &= \ln(2) - \ln(1/2) = \ln \left(\frac{2}{1/2} \right) = \ln(4). \end{aligned}$$

Alternatively, one could compute the length of γ_2 by using the equation of the circle, $x^2 + y^2 = 5^2 \implies y = \sqrt{5^2 - x^2}$, thus $\frac{dy}{dx} = \frac{-x}{\sqrt{5^2 - x^2}}$, and use the formula $ds = \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{y} dx$. As we have $-3 \leq x \leq 3$, we get the length of γ_2 to be

$$\begin{aligned} \int_{-3}^3 \frac{\sqrt{1 + \frac{x^2}{5^2 - x^2}}}{\sqrt{5^2 - x^2}} dx &= \int_{-3}^3 \frac{\sqrt{\frac{5^2}{5^2 - x^2}}}{\sqrt{5^2 - x^2}} dx \\ &= \int_{-3}^3 5 \cdot \frac{1}{5^2 - x^2} dx \\ &= 5 \int_{-3}^3 \frac{\frac{1}{10}}{5 - x} + \frac{\frac{1}{10}}{5 + x} dx \\ &= \frac{5}{10} (-\ln(5 - x) + \ln(5 + x)) \Big|_{-3}^3 \\ &= \frac{1}{2} (-\ln(2) + \ln(8) - (-\ln(8) + \ln(2))) \\ &= \ln(8) - \ln(2) = \ln\left(\frac{8}{2}\right) = \ln(4). \end{aligned}$$

This method can also be used to compute the length of γ_1 (you can do on your own).

- (c) By definition, the distance between two points is the minimum length of all possible paths from P to Q , and γ_2 is such a path of length $\ln(4)$, hence it follows immediately that the hyperbolic distance between P and Q is less than or equal to $\ln(4)$.

Problem 2. Let $a, b, c, d \in \mathbb{R}$ be given such that $ad - bc = 1$. Consider the map

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \quad f(z) = \frac{az + b}{cz + d}.$$

(a) Show that the restriction of f to \mathbb{H}^2 yields a map

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2.$$

That is, $\text{Im}(f(z)) > 0$ if $\text{Im}(z) > 0$.

(b) Show that f is a hyperbolic isometry.

(a) Suppose $z = \alpha + \beta i$, where $\alpha, \beta \in \mathbb{R}, \beta > 0$. Note that c, d can't both be 0, otherwise $ad - bc = 0 \neq 1$. So $(c\alpha + d)^2 + (c\beta)^2 > 0$ (this is to double check that we aren't accidentally dividing by 0 in the following computations), and we observe:

$$\begin{aligned} f(z) &= \frac{a(\alpha + \beta i) + b}{c(\alpha + \beta i) + d} \\ &= \frac{a\alpha + b + a\beta i}{c\alpha + d + c\beta i} \cdot \frac{(c\alpha + d) - c\beta i}{(c\alpha + d) - c\beta i} \\ &= \frac{(a\alpha + b)(c\alpha + d) + a\beta i(c\alpha + d) - c\beta i(a\alpha + b) + ac\beta^2}{(c\alpha + d)^2 + (c\beta)^2} \\ &= \frac{(a\alpha + b)(c\alpha + d) + ac\beta^2}{(c\alpha + d)^2 + (c\beta)^2} + \frac{ac\alpha\beta i + ad\beta i - ac\alpha\beta i - bc\beta i}{(c\alpha + d)^2 + (c\beta)^2} \\ &= \frac{(a\alpha + b)(c\alpha + d) + ac\beta^2}{(c\alpha + d)^2 + (c\beta)^2} + \frac{(ad - bc)\beta i}{(c\alpha + d)^2 + (c\beta)^2} \\ &= \frac{(a\alpha + b)(c\alpha + d) + ac\beta^2}{(c\alpha + d)^2 + (c\beta)^2} + \frac{\beta}{(c\alpha + d)^2 + (c\beta)^2} i \end{aligned}$$

since $\beta > 0$ and $(c\alpha + d)^2 + (c\beta)^2 > 0$, we have $\text{Im}(f(z)) = \frac{\beta}{(c\alpha + d)^2 + (c\beta)^2} > 0$, hence $f(z) \in \mathbb{H}^2$.

(b) By the discussions in section 4.2, we have that the following functions are isometries on \mathbb{H}^2 : $t_\alpha(z) = z + \alpha, \alpha \in \mathbb{R}, d_\rho(z) = \rho z, \rho \in \mathbb{R}_{>0}, \bar{r}(z) = -\bar{z}$. In addition, the inversion $I(z) = \frac{1}{\bar{z}}$ was also shown to be an isometry in section 4.2. Now, we consider the following functions on \mathbb{H}^2 : $A(z) = c^2 z + cd, B(z) = -\frac{1}{z}, C(z) = z + \frac{a}{c}$. Note that $A(z) = t_{cd} \circ d_{c^2}(z), \bar{r} \circ I(z) = \bar{r}(\frac{1}{\bar{z}}) = \bar{r}(\frac{z}{\bar{z}z}) = \bar{r}(\frac{z}{|z|^2}) = -\frac{\bar{z}}{|z|^2} = -\frac{\bar{z}}{\bar{z}z} = -\frac{1}{z} = B(z)$, and $C(z) = t_{\frac{a}{c}}(z)$. In particular, A, B, C are all products of isometries, hence are isometries themselves. Finally, we

observe:

$$\begin{aligned}
 C \circ B \circ A(z) &= C \circ B(c^2z + cd) \\
 &= C\left(-\frac{1}{c(cz + d)}\right) \\
 &= -\frac{1}{c(cz + d)} + \frac{a}{c} \\
 &= -\frac{ad - bc}{c(cz + d)} + \frac{a(cz + d)}{c(cz + d)} \\
 &= \frac{acz + ad - ad + bc}{c(cz + d)} \\
 &= \frac{az + b}{cz + d} = f(z).
 \end{aligned}$$

Thus, f is a composition of hyperbolic isometries, hence f itself is a hyperbolic isometry.

Alternatively, we can show this using the distance $ds = \frac{|dz|}{\text{Im}(z)}$. So letting $w = f(z) = \frac{az+b}{cz+d}$, we want to show $\frac{|dw|}{\text{Im}(w)} = \frac{|dz|}{\text{Im}(z)}$. By the chain and quotient rule, we compute $|dw| = \left| \frac{a(cz+d) - c(az+b)}{(cz+d)^2} dz \right| = \left| \frac{ad-bc}{(cz+d)^2} \right| |dz| = \frac{1}{(cz+d)^2} |dz|$. We also have $w = \frac{(az+b)(\overline{cz+d})}{(cz+d)(\overline{cz+d})} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{acz\bar{z} + bd + adz + bc\bar{z}}{|cz+d|^2} = \frac{ac|z|^2 + bd + (ad+bc)\text{Re}(z) + (ad-bc)\text{Im}(z)}{|cz+d|^2}$, and thus $\text{Im}(w) = \frac{ad-bc}{|cz+d|^2} \text{Im}(z) = \frac{1}{|cz+d|^2} \text{Im}(z)$, and it follows that $\frac{|dw|}{\text{Im}(w)} = \frac{|dz|}{\text{Im}(z)}$.

Problem 3. (25 points) Consider the hyperbolic isometry

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{1}{\bar{z}}.$$

(a) Compute the image of the points of the form ki and $\frac{i}{k}$ where $k \in \mathbb{N}$.

(b) Show that all the points of the unit half-circle

$$S := \{z \in \mathbb{H}^2 : |z| = 1\}$$

are fixed under f .

(c) Prove that S exactly coincides with the set of fixed points of f .

(a) We observe that $f(ki) = \frac{1}{-ki} = \frac{i}{k}$, and $f\left(\frac{i}{k}\right) = \frac{1}{-i/k} = \frac{k}{-i} = ki$.

(b) Let $z = \alpha + \beta i \in \mathbb{H}^2$ such that $|z| = \sqrt{\alpha^2 + \beta^2} = 1$. It follows that $\alpha^2 + \beta^2 = 1$, and we observe that $f(z) = \frac{1}{\alpha - \beta i} \cdot \frac{\alpha + \beta i}{\alpha + \beta i} = \frac{\alpha + \beta i}{\alpha^2 + \beta^2} = \alpha + \beta i = z$, so all the points of S are fixed under f .

(c) Part (b) shows one direction, and it remains to show that if a point z is fixed by f , then $z \in S$. So assume $f(z) = z$. Then $z = f(z) = \frac{1}{\bar{z}} = \frac{z}{|z|^2}$, which implies $|z|^2 = 1$, and hence $|z| = 1$, therefore $z \in S$.

Problem 4. (25 points) Consider the family of hyperbolic isometries

$$(2.1) \quad f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$ be given such that $ad - bc = 1$.

- (a) Let $P, Q \in \mathbb{H}^2$ be two points, show that there exists a hyperbolic isometry of the form (2.1) above such that $f(P) = Q$.
- (b) Let $P \in \mathbb{H}^2$ be an arbitrary point. Describe all the isometries of the form (2.1) such that $f(P) = P$.

- (a) Without loss of generality, we may assume $P = i$. (If not, we may apply a translation to move P to the y -axis, then a dilation to scale it to i). Let $Q = \gamma + \delta i$. We want to find an $f(z) = \frac{az+b}{cz+d}$ such that $f(P) = Q$, so we compute the following, noting that there is a choice we're allowed to make:

$$\begin{aligned} f(P) &= \frac{ai + b}{ci + d} \cdot \frac{d - ci}{d - ci} = \frac{ac + bd + (ad - bc)i}{c^2 + d^2} = Q = \gamma + \delta i \\ \implies \delta &= \frac{ad - bc}{c^2 + d^2} = \frac{1}{c^2 + d^2}, \text{ we choose } d = 0, c = \frac{1}{\sqrt{\delta}}. \end{aligned}$$

$$\text{We also have } \gamma = \frac{ac + bd}{c^2 + d^2} = \frac{\frac{a}{\sqrt{\delta}}}{\frac{1}{\delta}} = \frac{a\delta}{\sqrt{\delta}} = a\sqrt{\delta} \implies a = \frac{\gamma}{\sqrt{\delta}}.$$

$$\text{Finally, we must have } 1 = ad - bc = -\frac{b}{\sqrt{\delta}} \implies b = -\sqrt{\delta}.$$

So we take $a = \frac{\gamma}{\sqrt{\delta}}, b = -\sqrt{\delta}, c = \frac{1}{\sqrt{\delta}}, d = 0$. Note that $\delta > 0$, so $a, b, c, d \in \mathbb{R}$, and we check that $ad - bc = 0 + 1 = 1$, and $f(P) = \frac{\frac{\gamma}{\sqrt{\delta}}i - \sqrt{\delta}}{\frac{1}{\sqrt{\delta}}i} = \frac{-\sqrt{\delta}i - \frac{\gamma}{\sqrt{\delta}}}{-1/\sqrt{\delta}} = \gamma + \delta i = Q$, as desired.

Another, perhaps easier way, is to construct f as a composition of known isometries, namely translations and dilations. Let $P = \alpha + \beta i, Q = \gamma + \delta i$, with $\alpha, \beta, \gamma, \delta \in \mathbb{R}, \beta, \delta > 0$. We first apply the translation $t_{-\alpha}(z) = z - \alpha$ to send P to βi , then apply the dilation $d_{\delta/\beta}(z) = \frac{\delta z}{\beta}$ to send βi to δi , and finally apply the translation $t_{\gamma}(z) = z + \gamma$ to send δi to $\gamma + \delta i = Q$. Note that

the matrix representation of translation by α is $t_{\alpha}(z) = \frac{z+\alpha}{0+1} \longleftrightarrow \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ and

dilation by ρ is $d_{\rho}(z) = \frac{\rho z + 0}{0 + \frac{1}{\rho}} \longleftrightarrow \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & \frac{1}{\sqrt{\rho}} \end{bmatrix}$. Looking at the matrix

representations of these, we have that

$$\begin{aligned}
 f = t_\gamma \circ d_{\delta/\beta} \circ t_{-\alpha} &\longleftrightarrow \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\delta}{\beta}} & 0 \\ 0 & \sqrt{\frac{\beta}{\delta}} \end{bmatrix} \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\delta}{\beta}} & -\alpha\sqrt{\frac{\delta}{\beta}} \\ 0 & \sqrt{\frac{\beta}{\delta}} \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{\frac{\delta}{\beta}} & -\alpha\sqrt{\frac{\delta}{\beta}} + \gamma\sqrt{\frac{\beta}{\delta}} \\ 0 & \sqrt{\frac{\beta}{\delta}} \end{bmatrix}.
 \end{aligned}$$

So the desired isometry sending P to Q is

$$f(z) = \frac{\sqrt{\frac{\delta}{\beta}}z - \alpha\sqrt{\frac{\delta}{\beta}} + \gamma\sqrt{\frac{\beta}{\delta}}}{0z + \sqrt{\frac{\beta}{\delta}}}.$$

- (b) We first investigate the fixed points of an arbitrary isometry $f(z) = \frac{az+b}{cz+d}$ (assuming $f \neq \text{Id}$, since the identity fixes all points). Note that if z_0 is a fixed point of f , then

$$\begin{aligned}
 z_0 = f(z_0) &= \frac{az_0 + b}{cz_0 + d} \implies cz_0^2 + (d - a)z_0 - b = 0 \\
 \implies z_0 &= \frac{a - d \pm \sqrt{(d - a)^2 + 4bc}}{2c} \\
 &= \frac{a - d \pm \sqrt{a^2 + d^2 - 2ad + 4bc}}{2c} \\
 &= \frac{a - d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2c} \\
 &= \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}.
 \end{aligned}$$

If $(a + d)^2 - 4 \geq 0$, then the two possible fixed points are real (i.e. $\text{Im}(z_0) = 0$, hence $z_0 \notin \mathbb{H}^2$). If $(a + d)^2 - 4 < 0$, then there are two possible fixed points, but $\frac{a-d-\sqrt{(a+d)^2-4}}{2c}$ has a negative imaginary part, hence is also not in \mathbb{H}^2 . In particular, any such isometry can have at most one fixed point in \mathbb{H}^2 . So if $P = \alpha + \beta i$ is an arbitrary point in \mathbb{H}^2 , then the isometries of the form (2.1) that fix P satisfy $\frac{a-d}{2c} = \alpha$, $\frac{\sqrt{4-(a+d)^2}}{2c} = \beta$, and $ad - bc = 1$, $|a + d| < 2$. However, we do note that this probably doesn't give us much intuition on what these isometries look like, so let us also give a geometric description of them. Note that isometries of the form (2.1) are orientation-preserving isometries by definition, and by the hyperbolic three reflections theorem, we know that such isometries must be products of two reflections. Certainly, reflections across two disjoint lines (geodesics) have no fixed points (reflections across ultraparallel lines, i.e. actually disjoint lines, are translations along a line, and reflections across asymptotic lines, i.e. lines that meet at the boundary of \mathbb{H}^2 , are rotations about the common limit point). So the remaining case would be reflections across intersecting lines, which are rotations about the point of intersection. Therefore,

the isometries of the form (2.1) that fix an arbitrary point $P \in \mathbb{H}^2$ are products of reflections across two lines intersecting at the point P , and equivalently, these are rotations of \mathbb{H}^2 about P .

Remark. It is not necessary to simplify the computations of the fixed point the way we did here. But rewriting in this form is nice because if we identify $f \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $(a + d)^2$ is precisely the square of the trace of the matrix, which would be helpful when we are classifying Möbius Transformations. The ones we found here are called *elliptic*, and the other two types of Möbius Transformations are *parabolic* and *hyperbolic*.

Problem 5. (25 points) Let $a, b, c, d \in \mathbb{R}$ be given such that $ad - bc = 1$. Consider the maps

$$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad f(z) = \frac{-1}{z}.$$

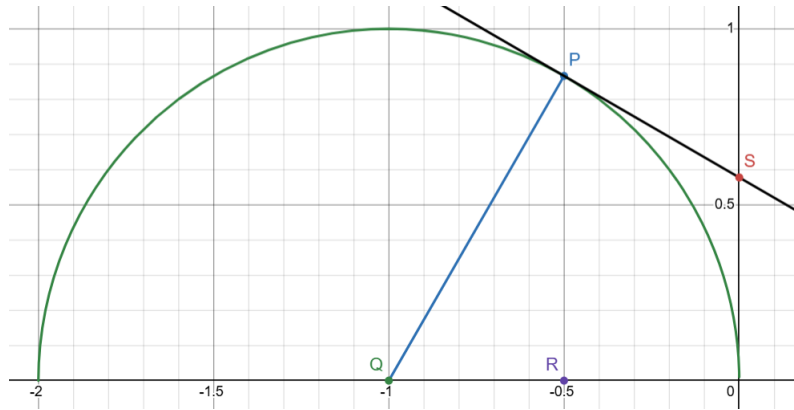
$$g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad g(z) = \frac{-1}{z+1}.$$

- (a) Find the fixed points of f and g .
- (b) Describe f and g geometrically as well as you can.
- (c) Show that $f^2 := f \circ f$ is the identity.
- (d) Show that $g^3 := g \circ g \circ g$ is the identity.
- (g) Consider the composition $h = f \circ g$. Describe h geometrically and rigorously show that there exists no $n \in \mathbb{N}$ such that h^n is the identity.

- (a) If z is a fixed point of f , then we must have $z = -\frac{1}{z} \implies z^2 = -1 \implies z = i$. If z is a fixed point of g , then $z = -\frac{1}{z+1} \implies z^2 + z + 1 = 0 \implies z = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$, but we only take the one with positive imaginary part. Therefore, the fixed point of f is i , and the fixed point of g is $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

- (b) By 4(b), f, g are both rotations about their fixed point. We try to determine the angle of rotation. For f , we observe that, say $2i$, is sent to $\frac{-1}{2i} = \frac{i}{2}$, so $2i, \frac{i}{2}$ are on the same (vertical) geodesic, thus f is a rotation about i at angle π . For g , we look at points of the form $-\frac{1}{2} + \beta i$ for $\beta > 0$, which lie on the vertical geodesic through the fixed point. Then, we see that $g(-\frac{1}{2} + \beta i) = -\frac{1}{-\frac{1}{2} + \beta i} = -\frac{\frac{1}{2} - \beta i}{\frac{1}{4} - \beta^2} = \frac{-2}{1 + 4\beta^2} + \frac{4\beta}{1 + 4\beta^2}i$, and this maps out the geodesic with equation $(x+1)^2 + y^2 = 1$. We want to find the angle this geodesic makes with the vertical line $x = -\frac{1}{2}$ (which is the angle between the tangent lines). This is probably overkill, but here's a construction without using any crazy machinery:

By construction, we know \overline{QP} makes an angle of $\pi/3$ with the x -axis, thus $\angle QPR = \pi/6$. By definition, the tangent line of a circle is perpendicular to the radius of the circle meeting the point of tangency, hence $\angle SPQ = \pi/2$ (S is



any point on the tangent line to the right of P). It follows that $\angle SPR = \pi/3$, and its supplementary angle is $2\pi/3$. Now, we need to identify whether g is a rotation by $\pi/3$ or by $-2\pi/3$ (rotation counterclockwise vs clockwise). Take the point $-\frac{1}{2} + i$, which is above P . We observe that $g(-\frac{1}{2} + i) = -\frac{2}{5} + \frac{4}{5}i$, which is a point to the right of P . Thus, we deduce that g is a rotation about $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ by angle $-2\pi/3$ (or equivalently, by angle $4\pi/3$).

(c) We compute $f \circ f(z) = f(-\frac{1}{z}) = \frac{-1}{-1/z} = z$, so $f^2 = \text{Id}$.

(d) We compute

$$\begin{aligned} g \circ g \circ g(z) &= g \circ g \left(\frac{-1}{z+1} \right) \\ &= g \left(\frac{-1}{\frac{-1}{z+1} + 1} \right) \\ &= g \left(\frac{-1}{\frac{z}{z+1}} \right) \\ &= g \left(-\frac{z+1}{z} \right) \\ &= \frac{-1}{\frac{-z-1}{z} + 1} \\ &= \frac{-1}{\frac{-1}{z}} = z, \end{aligned}$$

so $g^3 = \text{Id}$.

(e) We compute the formula for h directly: $h(z) = f \circ g(z) = f\left(\frac{-1}{z+1}\right) = \frac{-1}{\frac{-1}{z+1}} = z+1$, which is a limit rotation about ∞ , $t_1(z) = 1+z$. We show by induction that $h^n(z) = z+n$. The basis for induction, $n=1$, is obvious by definition. So assume $h^k(z) = z+k$ for some $k \in \mathbb{N}$. Then $h^{k+1}(z) = h \circ h^k(z) = h(z+k) = z+k+1$, as desired. Since $n \neq 0$ for all $n \in \mathbb{N}$, it follows that there is no $n \in \mathbb{N}$ such that $h^n = \text{Id}$.