# MAT 141: PROBLEM SET 5

DUE TO FRIDAY MAY 22 AT 11:00AM

Abstract. This is the fourth problem set for the Euclidean and Non-Euclidean Geometry Course in the Spring Quarter 2026. It is due Friday May 15 at 11:00am via online submission.

## 1. Instructions

**Purpose**: The goal of this assignment is to practice problems on isometries and lines in the Hyperbolic Plane $\mathbb{H}^2$.

**Task**: Solve Problems 1 through 5 below. Problems 1 and 6 will not be graded but I trust that you will work on it. Problems 2,3,4 and 5 will be graded.

**Instructions**: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

**Grade**: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

**Textbook**: We will use "Geometry of Surfaces" by J. Stillwell.

**Writing**: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

**Mathematical commments**: Every instance of "length" or "distance" refers to *hyperbolic lengths* and *hyperbolic distance*. Similarly, all isometries are taken to be hyperbolic isometries. We use the notation $z \in \mathbb{H}^2$ to indicate complex coordinates in the hyperbolic upper-half plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

## 2. Problems

**Problem 1**. For each of the following points $Q \in \mathbb{H}^2$, find an equation for the unique hyperbolic line $L \subseteq \mathbb{H}^2$ equidistant to $P = i$ and $Q$:

$$Q = 2 + i, \qquad Q = 2i, \qquad Q = 2i + 3.$$

**Problem 2**. (25 points) Let $P \in \mathbb{H}^2$ be a point in the hyperbolic plane and $L \subseteq \mathbb{H}^2$ a hyperbolic line containing $P$. Consider a point $Q \in \mathbb{H}^2$ outside of $L$. Show that there exists infinitely many hyperbolic lines through $Q$ that are parallel to $L$, i.e. that do not intersect $L$.

**Problem 3**. (25 points) Consider the hyperbolic rotation $\varphi : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ around $i$ of angle $2\pi/5$.

(a) Write a formula for $\varphi$ in the form of

$$f(z) = \frac{az + b}{cz + d}.$$

(b) Describe equations for the images of the line $L = \{z \in \mathbb{H}^2 : |z| = 1$ under each of $\varphi^k$ for $k = 1, 2, 3, 4$, and draw them.

(c) Describe equations for the images of the line $M = \{z \in \mathbb{H}^2 : \mathrm{Re}(z) = 2\}$ under each of $\varphi^k$ for $k = 1, 2, 3, 4$, and draw them.

**Problem 4**. (25 points) Consider the following three hyperbolic lines

$$L = \{z \in \mathbb{H}^2 : |z| = 1\}, M = \{z \in \mathbb{H}^2 : \mathrm{Re}(z) = 0\}, N = \{z \in \mathbb{H}^2 : |z| = 2\}.$$

(a) Describe the composition $f = r_N \circ r_M \circ r_L$ as

$$f(z) = \frac{-a\bar{z} + b}{-c\bar{z} + d}.$$

(b) Show that there are no fixed points for $f$.

(c) Find any hyperbolic lines $S \subseteq \mathbb{H}^2$ such that $f(S) = S$.

(d) Draw the images of $L$ under the iterates $f, f^2, f^3$ of $f$, providing equations for these lines $f(L), f^2(L)$ and $f^3(L)$.

(e) Draw the images of the line $\{z \in \mathbb{H}^2 : |z - 5| = 0.5\}$ under the iterates $f, f^2, f^3, f^4$ of $f$.

**Problem 5**. (25 points) Consider the subset

$$F := \{z \in \mathbb{H}^2 : |z| \geq 1, |\mathrm{Re}(z)| \leq 0.5\}.$$

Show that any point $P \in \mathbb{H}^2$ can be brought to a unique point in $F$ under an isometry $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ of the form

$$f(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.$$

(The key point of this exercise is that we force $a, b, c, d \in \mathbb{Z}$, not $a, b, c, d \in \mathbb{R}$.)

**Problem 6**. Solve the following parts:

(a) Show that there is a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$ for positive integers $p, q, r \in \mathbb{N}$ if and only if
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

(b) Find the hyperbolic triangle with $\pi/p, \pi/q, \pi/r$, for some positive integers $p, q, r \in \mathbb{N}$, with smallest area.

(c) Show that you can tile the hyperbolic plane $\mathbb{H}^2$ with triangles of angles $\pi/2, \pi/3, \pi/7$.

(d) (Challenging) Describe the hyperbolic isometries that preserve the tiling in ($c$).