# MAT 141: PROBLEM SET 5

DUE TO FRIDAY MAY 22 AT 11:00AM

Abstract. This is the fourth problem set for the Euclidean and Non-Euclidean Geometry Course in the Spring Quarter 2026. It is due Friday May 15 at 11:00am via online submission.

## 1. Instructions

**Purpose**: The goal of this assignment is to practice problems on isometries and lines in the Hyperbolic Plane $\mathbb{H}^2$.

**Task**: Solve Problems 1 through 5 below. Problems 1 and 6 will not be graded but I trust that you will work on it. Problems 2,3,4 and 5 will be graded.

**Instructions**: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

**Grade**: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

**Textbook**: We will use "Geometry of Surfaces" by J. Stillwell.

**Writing**: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

**Mathematical commments**: Every instance of "length" or "distance" refers to *hyperbolic lengths* and *hyperbolic distance*. Similarly, all isometries are taken to be hyperbolic isometries. We use the notation $z \in \mathbb{H}^2$ to indicate complex coordinates in the hyperbolic upper-half plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

## 2. PROBLEMS

**Problem 1**. For each of the following points $Q \in \mathbb{H}^2$, find an equation for the unique hyperbolic line $L \subseteq \mathbb{H}^2$ equidistant to $P = i$ and $Q$:

$$Q = 2 + i, \qquad Q = 2i, \qquad Q = 2i + 3.$$

The general method for finding the hyperbolic line through two points is to construct the perpendicular bisector of the two points, and take the half circle through the two points, centered at the intersection of the perpendicular bisector with the $x$-axis. If there is no such intersection, then the two points are in the same vertical line, which is the geodesic through the two points. So for $P = i, Q = 2 + i$, the perpendicular bisector is $x = 1$, and the distance from $(1, 0)$ to $P$ is $\sqrt{2}$, thus the hyperbolic line through $P, Q$ is $(x - 1)^2 + y^2 = 2$.
For $P = i, Q = 2i$, they lie on the same vertical line, hence the hyperbolic line through them is $x = 0$.
For $P = i, Q = 2i + 3$, the slope from $P$ to $Q$ is $\frac{1}{3}$, so the perpendicular bisector has slope $-3$, and passes through the midpoint $(\frac{3}{2}, \frac{3}{2})$, thus the perpendicular bisector is $y - \frac{3}{2} = -3(x - \frac{3}{2})$. When $y = 0$, we get $-\frac{3}{2} = -3x + \frac{9}{2} \implies -3x = -6 \implies x = 2$. The distance from $(2, 0)$ to $P$ is $\sqrt{5}$, so the equation for the hyperbolic line through $P, Q$ is $(x - 2)^2 + y^2 = 5$.

**Problem 2**. (25 points) Let $P \in \mathbb{H}^2$ be a point in the hyperbolic plane and $L \subseteq \mathbb{H}^2$ a hyperbolic line containing $P$. Consider a point $Q \in \mathbb{H}^2$ outside of $L$. Show that there exists infinitely many hyperbolic lines through $Q$ that are parallel to $L$, i.e. that do not intersect $L$.
Let $P = \alpha + \beta i, Q = \gamma + \delta i$. By conjugation, we may assume without loss of generality that $L$ is the vertical line through $P$, and that $\alpha < \gamma$. (Otherwise, we conjugate by a rotation to make the line vertical, and so that $Q$ is to the right of the line). We consider points on $\partial\mathbb{H}^2$ of the form $(t, 0)$, for $\alpha < t < \gamma$. We know by construction that there is a unique line through $Q$ and each $(t, 0)$. By continuity of $\mathbb{R}$, we know that there exists infinitely many real numbers between $\alpha$ and $\gamma$, and hence we have constructed infinitely many lines through $Q$. It remains to show that all of these lines are parallel to $L$. Since $t < \gamma$, we see that the perpendicular bisector of $(t, 0)$ and $Q$ will intersect the $x$-axis at some point $(s, 0)$ with $s > t$, thus the left most point of any geodesic through $(t, 0)$ and $Q$ is $(t, 0)$, thus these geodesics will not intersect $L$.

**Problem 3**. (25 points) Consider the hyperbolic rotation $\varphi : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ around $i$ of angle $2\pi/5$.

(a) Write a formula for $\varphi$ in the form of

$$f(z) = \frac{az + b}{cz + d}.$$

(b) Describe equations for the images of the line $L = \{z \in \mathbb{H}^2 : |z| = 1\}$ under each of $\varphi^k$ for $k = 1, 2, 3, 4$, and draw them.

(c) Describe equations for the images of the line $M = \{z \in \mathbb{H}^2 : \mathrm{Re}(z) = 2\}$ under each of $\varphi^k$ for $k = 1, 2, 3, 4$, and draw them.

(a) In case the rotation formula hadn't been derived in class yet, let us derive it here. In sections 4.4, 4.5 of the textbook, Stillwell gave us the orientation-preserving $\mathbb{D}^2$-isometries are of the form $(*) : f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, where $a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1$; the rotation about 0 by angle $\theta$ in $\mathbb{D}^2$ is given by $R_\theta(z) = e^{i\theta}z$; and $\mathbb{D}^2$-isometries are the functions $JhJ^{-1}$, where $h$ is an $\mathbb{H}^2$-isometry, and $J = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$, $J^{-1} = \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$. Note that $J$ sends $i \in \mathbb{H}^2$ to $0 \in \mathbb{D}^2$. So first, we write $R_\theta(z)$ in the form $(*)$, then conjugate to get $J^{-1}R_\theta J$, which is an $\mathbb{H}^2$-isometry, and this gives us the rotation about $i$ by angle $\theta$. So, we have that $R_\theta(z) = e^{i\theta}z = \frac{e^{i\theta}z+0}{0z+1}$, but we want the diagonal to be $a, \bar{a}$, so we rewrite to get $R_\theta(z) = \dfrac{e^{i\frac{\theta}{2}}z+0}{0z+e^{-i\frac{\theta}{2}}}$. Then, we compute:

$$\begin{aligned}
&J^{-1} \circ R_\theta \circ J(z) \\
&= \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{bmatrix} \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \\
&= \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} ie^{i\frac{\theta}{2}} & e^{i\frac{\theta}{2}} \\ e^{-i\frac{\theta}{2}} & ie^{-i\frac{\theta}{2}} \end{bmatrix} \\
&= \begin{bmatrix} e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} & -ie^{i\frac{\theta}{2}} + ie^{-i\frac{\theta}{2}} \\ ie^{i\frac{\theta}{2}} - ie^{-i\frac{\theta}{2}} & e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} + \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & -i(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) + i(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}) \\ i(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) - i(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}) & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} + \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} \end{bmatrix} \\
&= \begin{bmatrix} 2\cos\frac{\theta}{2} & 2\sin\frac{\theta}{2} \\ -2\sin\frac{\theta}{2} & 2\cos\frac{\theta}{2} \end{bmatrix},
\end{aligned}$$

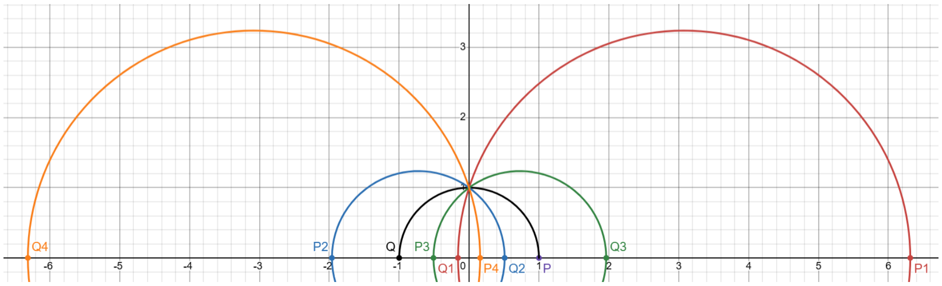
and finally, we normalize so that the determinant is 1, and thus we have that the rotation about $i$ by $\theta$ in $\mathbb{H}^2$ is

$$\begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}.$$

Hence, $\varphi(z) = \frac{\cos(\pi/5)z+\sin(\pi/5)}{-\sin(\pi/5)z+\cos(\pi/5)}$.

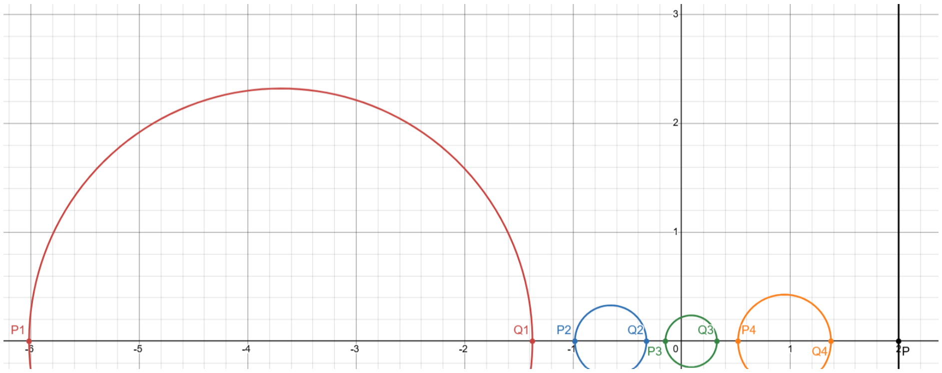
(b) One way to get the equations of the images of the lines is to see how the endpoints of $L$ are mapped, and find the corresponding geodesic through the images of the endpoints (i.e. compute the midpoint, which still lies on the $x$-axis, and take the circle centered at the midpoint with radius half the distance between the two endpoints). The equations will not look pretty, though if you know how to type this up using some coding, that would be sweet. But we can also just do this by brute force. Let $P = 1, Q = -1$ be the two initial endpoints. Define $P_k = \varphi^k(P), Q_k = \varphi^k(Q)$ for $k = 1, 2, 3, 4$. We have $P_1 := \varphi(P) = \frac{\cos(\pi/5)+\sin(\pi/5)}{-\sin(\pi/5)+\cos(\pi/5)} \approx 6.31375151, Q_1 := \varphi(Q) = \frac{-\cos(\pi/5)+\sin(\pi/5)}{\sin(\pi/5)+\cos(\pi/5)} \approx -0.15838444$. The midpoint is $\frac{P_1+Q_1}{2} \approx \frac{6.31375151-0.15838444}{2} = 3.07768353718$, and the radius of the corresponding geodesic through $P_1, Q_1$ is $\frac{6.31375151+0.15838444}{2}$, and thus

$\varphi(L) = \{(x,y) \in \mathbb{H}^2 : (x-3.07768353718)^2+y^2 = (\frac{6.31375151+0.15838444}{2})^2\}$. In the same way, we construct $\varphi^k(L) = \{(x,y) \in \mathbb{H}^2 : (x - \frac{P_k+Q_k}{2})^2 + y^2 = (\frac{P_k-Q_k}{2})^2\}$, and perhaps it isn't really helpful to explicitly write these out numerically, but here's the picture:



The black line is $L$, the red line is $\varphi(L)$, the blue line is $\varphi^2(L)$, the green line is $\varphi^3(L)$, and the orange line is $\varphi^4(L)$.

(c) We do the same thing as (b). Let $P = 2, Q = \infty$, and define $P_k := \varphi^k(P), Q_k := \varphi^k(Q)$. First note that $Q_1 = \varphi(\infty) = \frac{\cos(\pi/5)}{-\sin(\pi/5)}$ (basically, just plug $\infty$ into the function. If you want to be more rigorous, you can plug in a variable and take the limit as the variable approaches $\infty$). The other parts are exactly the same, so we have $\varphi^k(L) = \{(x,y) \in \mathbb{H}^2 : (x - \frac{P_k+Q_k}{2})^2 + y^2 = (\frac{P_k-Q_k}{2})^2\}$, and they look like:



with the same coloring as (b), i.e. the black line is $M$, the red line is $\varphi(M)$, the blue line is $\varphi^2(M)$, the green line is $\varphi^3(M)$, and the orange line is $\varphi^4(M)$.

**Problem 4**. (25 points) Consider the following three hyperbolic lines

$$L = \{z \in \mathbb{H}^2 : |z| = 1\}, M = \{z \in \mathbb{H}^2 : \mathrm{Re}(z) = 0\}, N = \{z \in \mathbb{H}^2 : |z| = 2\}.$$

(a) Describe the composition $f = r_N \circ r_M \circ r_L$ as

$$f(z) = \frac{-a\bar{z}+b}{-c\bar{z}+d}.$$

(b) Show that there are no fixed points for $f$.

(c) Find any hyperbolic lines $S \subseteq \mathbb{H}^2$ such that $f(S) = S$.

(d) Draw the images of $L$ under the iterates $f, f^2, f^3$ of $f$, providing equations for these lines $f(L), f^2(L)$ and $f^3(L)$.

(e) Draw the images of the line $\{z \in \mathbb{H}^2 : |z-5| = 0.5\}$ under the iterates $f, f^2, f^3, f^4$ of $f$.

(a) First note that $r_L$ is the inversion in the unit circle, so $r_L(z) = \frac{1}{\overline{z}}$, and $r_M$ is the reflection across the $y$-axis, so $r_M(z) = -\overline{z}$ (these are given in section 4.2 of the textbook). For $r_N$, we conjugate by the dilation $d(z) = 2z, d^{-1}(z) = \frac{1}{2}z$, so that $r_N = d \circ r_L \circ d^{-1}$. Next, we write all of these in the form $\begin{bmatrix} -a & b \\ -c & d \end{bmatrix}$ for $a, b, c, d \in \mathbb{R}, ad - bc = 1$.

$$r_L(z) = \frac{1}{\overline{z}} \implies r_L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$r_M(z) = -\overline{z} \implies r_M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$r_N(z) = d \circ r_L \circ d^{-1}(z) = d \circ r_L(z/2) = d(2/\overline{z}) = \frac{4}{\overline{z}}$$

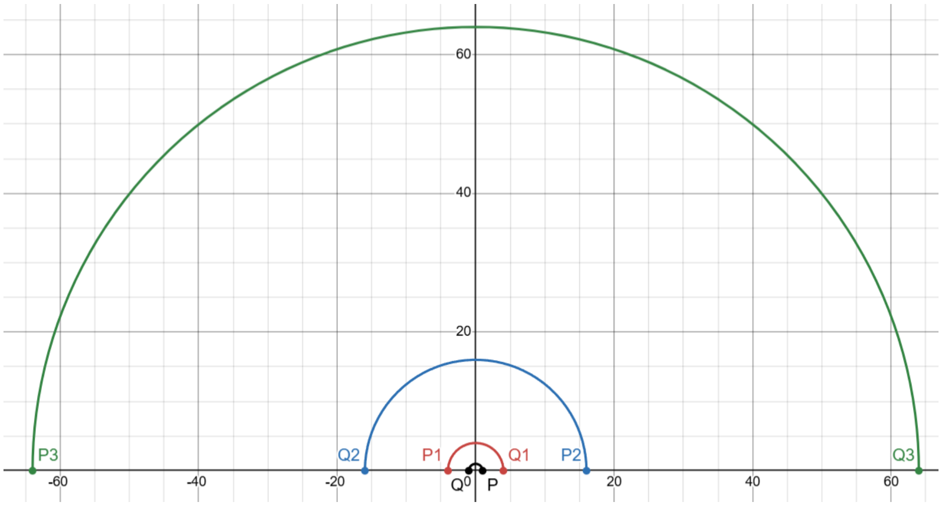
$$\implies r_N = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Then, $f = r_N \circ r_M \circ r_L = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$. So $f(z) = \frac{2\overline{z}+0}{0\overline{z}-\frac{1}{2}}$.

(b) If $z = \alpha + \beta i \in \mathbb{H}^2$ is a fixed point of $f$, then we have $z = \frac{2\overline{z}}{-\frac{1}{2}} \implies z = -4\overline{z} \implies \alpha + \beta i = -4\alpha + 4\beta i \implies \alpha = 0, \beta = 0 \implies z = 0 \notin \mathbb{H}^2$. Therefore, $f$ has no fixed points.

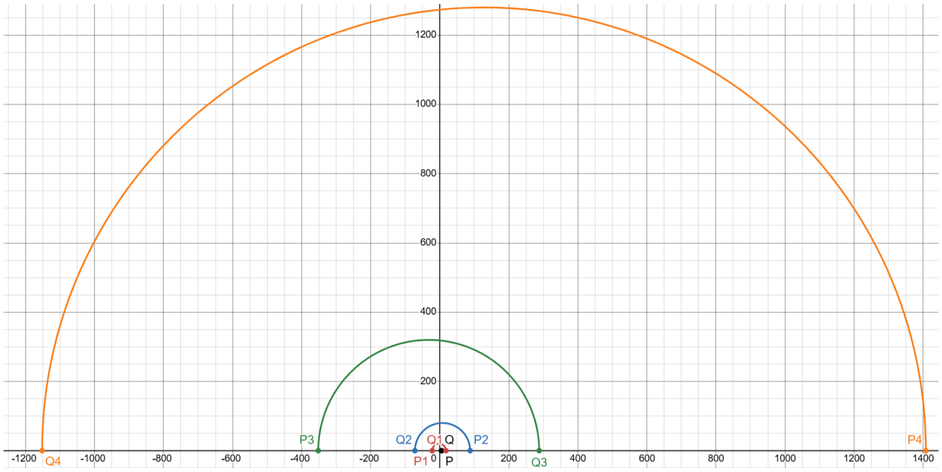
(c) Observe that $r_M \circ r_N \circ r_L = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, which we can normalize by $-1$ to get the same matrix as $f$. Then, we note that by the discussions in section 4.5 of the textbook, $r_N \circ r_L$ is some dilation $d_\rho(z) = \rho z$, which leaves invariant only the $y$-axis (it also leaves invariant curves of the form $y = kx$, where $k$ is a constant, but these are not lines in $\mathbb{H}^2$). Note also that $r_M$ is the reflection across the $y$-axis, hence it fixes the $y$-axis as well, and it follows that $S = \{(x, y) \in \mathbb{H}^2 : x = 0\}$ is a hyperbolic line such that $f(S) = S$ (one may also guess this as a fixed line, and check that the endpoints $0, \infty$ are both fixed by $f$). Now, we need to show that any other hyperbolic line is not fixed. To that end, it suffices to check the endpoints. We've already shown in part (b) that 0 is fixed under $f$, and it's also easy to see that $\infty$ is fixed. So we consider pairs of endpoints of the form $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. We have $f(\alpha) = \frac{2\alpha}{-\frac{1}{2}} = -4\alpha$, and likewise $f(\beta) = -4\beta$. If the geodesic through $\alpha, \beta$ is fixed by $f$, then we must have either $\alpha = -4\alpha, \beta = -4\beta$ (which is clearly impossible for $\alpha, \beta \in \mathbb{R} \setminus \{0\}$), or $\alpha = -4\beta, \beta = -4\alpha$. The latter case implies $\alpha = 16\alpha$, which is once again impossible as $\alpha \neq 0$. We conclude that the only fixed line is the $y$-axis.

(d) We do this in the same manner as 3b,3c. Let $P = 1, Q = -1$ be the initial endpoints of $L$. We define $P_k = f^k(P), Q_k = f^k(Q)$ for $k = 1, 2, 3$, and compute $P_1 = f(P) = -4, Q_1 = f(Q) = 4$, and likewise $P_2 = 16, Q_2 = -16, P_3 = -64, Q_3 = 64$. These pairs $P_k, Q_k$ have midpoint at 0, so the equations are $f(L) = \{(x, y) \in \mathbb{H}^2 : x^2 + y^2 = 4\}, f^2(L) = \{(x, y) \in \mathbb{H}^2 : x^2 + y^2 = 16\}, f^3(L) = \{(x, y) \in \mathbb{H}^2 : x^2 + y^2 = 64\}$.



The black line is $L$, the red line is $f(L)$, the blue line is $f^2(L)$, and the green line is $f^3(L)$.

(e) We do the same thing as part d, defining $P = 5.5, Q = 4.5, P_k = f^k(P), Q_k = f^k(Q)$ for $k = 1, 2, 3, 4$, so $P_1 = -22, Q_1 = 18, P_2 = 88, Q_2 = -72, P_3 = -352, Q_3 = 288, P_4 = 1408, Q_4 = -1152$, and the images of the given line $M = \{z \in \mathbb{H}^2 : |z - 5| = 0.5\} = \{(x, y) \in \mathbb{H}^2 : (x - 5)^2 + y^2 = \frac{1}{4}\}$, $f^k(M)$, are the geodesics through $P_k, Q_k$:



The black line is $L$, the red line is $f(M)$, the blue line is $f^2(M)$, the green line is $f^3(M)$, and the orange line is $f^4(M)$.

**Problem 5**. (25 points) Consider the subset

$$F := \{z \in \mathbb{H}^2 : |z| \geq 1, |\mathrm{Re}(z)| \leq 0.5\}.$$

Show that any point $P \in \mathbb{H}^2$ can be brought to a unique point in $F$ under an isometry $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.$$

(The key point of this exercise is that we force $a, b, c, d \in \mathbb{Z}$, not $a, b, c, d \in \mathbb{R}$.)

Note that $F$ is the triangle with vertices $-0.5 + \frac{\sqrt{3}}{2}i, 0.5 + \frac{\sqrt{3}}{2}i, \infty$. We first map everything to the vertical strip $S := \{z \in \mathbb{H}^2 : |\mathrm{Re}(z)| \leq 0.5\}$, and this is done by applying a suitable translation: every point $P$ is in some vertical strip $\alpha - 0.5 \leq x \leq \alpha + 0.5$, where $\alpha \in \mathbb{Z}$, so the translation $t_{-\alpha}(z) = z - \alpha = \frac{z-\alpha}{0+1}$ brings $P$ to $F$. Now, we assume $|P| < 1$. In particular, this implies $\mathrm{Im}(P) < 1$, and we apply the rotation about $i$ by $\pi$, $R_\pi(z) = -\frac{1}{z} = \frac{0z-1}{z+0}$ (we saw this in PSet 4). Observe that $\mathrm{Im}(R_\pi(P)) = \mathrm{Im}(-\frac{\overline{P}}{|P|^2}) = \frac{\mathrm{Im}(P)}{|P|^2} > \mathrm{Im}(P)$ because $|P| < 1$. This shows that $R_\pi$ strictly increases the imaginary part of $P$. If $P$ gets sent outside of $S$, then we apply a translation back to $S$, and repeat the process if $|P|$ is still less than 1. So in the orbit of such compositions of translations and rotations on $P$, we may pick $P'$ to be a point in $S$ with maximum imaginary part. If $|P'| < 1$, then applying the rotation gives us another point with larger imaginary part, which is a contradiction to $P'$ being the point with maximum imaginary part. Thus, we have $|P'| \geq 1$, hence $P' \in F$. This proves the existence part, and it remains to show uniqueness. In particular, we will show that any two distinct points in the interior of $F$ cannot be mapped to each other via an isometry of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{Z}, ad - bc = 1$. Assume that $z_1, z_2$ are distinct points in the interior of $F$, and $f(z_1) = \frac{az_1+b}{cz_1+d} = z_2$. We compare the imaginary parts of both sides. We have $\mathrm{Im}(z_2) = \mathrm{Im}(f(z_1)) = \mathrm{Im}\left(\frac{(az_1+b)(\overline{cz_1+d})}{|cz_1+d|^2}\right) = \frac{(ad-bc)\mathrm{Im}(z_1)}{|cz_1+d|^2} = \frac{\mathrm{Im}(z_1)}{|cz_1+d|^2}$. Writing $z_1 = \alpha + \beta i \in \mathbb{H}^2$, we have $|cz_1+d|^2 = (c\alpha+d)^2 + c^2\beta^2 = c^2\alpha^2 + 2cd\alpha + d^2 + c^2\beta^2$. Since $\alpha^2 + \beta^2 = |z_1|^2 \geq 1$, we see that $|cz_1 + d|^2 \geq c^2 + 2cd\alpha + d^2$. Now, because $z_1$ is in the interior of $F$, we have $|\alpha| = |\mathrm{Re}(z_1)| < \frac{1}{2}$, so $2cd\alpha > -|cd|$, and it follows that $|cz_1 + d|^2 > c^2 - |cd| + d^2 - |cd| + |cd| = (|c| - |d|)^2 + |cd|$. Since $c, d$ can't both be 0 (otherwise, we are dividing by 0), we must have either $(|c| - |d|)^2 > 0$ or $|cd| > 0$, and because $c, d \in \mathbb{Z}$, it follows that $|cz_1 + d|^2 > (|c| - |d|)^2 + |cd| \geq 1$. That is, $\mathrm{Im}(z_2) = \frac{\mathrm{Im}(z_1)}{|cz_1+d|^2} \implies \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$. By the same argument, $f^{-1}(z_2) = z_1 \implies \mathrm{Im}(z_2) < \mathrm{Im}(z_1)$, which is a contradiction. Thus, interior points of $F$ cannot be mapped to other interior points of $F$ under such isometries. the boundary of $F$, however, is not unique, as points on the line $x = -0.5$ can be mapped to points on the line $x = 0.5$ via the translation $t_1$.

**Problem 6**. Solve the following parts:

(a) Show that there is a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$ for positive integers $p, q, r \in \mathbb{N}$ if and only if

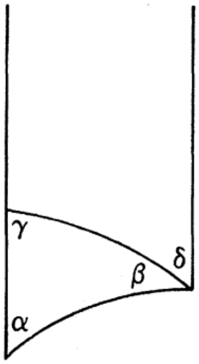
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

(b) Find the hyperbolic triangle with $\pi/p, \pi/q, \pi/r$, for some positive integers $p, q, r \in \mathbb{N}$, with smallest area.

(c) Show that you can tile the hyperbolic plane $\mathbb{H}^2$ with triangles of angles $\pi/2, \pi/3, \pi/7$.

(d) (Challenging) Describe the hyperbolic isometries that preserve the tiling in ($c$).

(a) By the corollary from section 4.7, we know that the area of a triangle with angles $\pi/p, \pi/q, \pi/r$ is $\pi - (\pi/p + \pi/q + \pi/r) = \pi(1 - (1/p + 1/q + 1/r))$. If such a triangle exists, then it must have positive area, and hence $1/p + 1/q + 1/r < 1$. Conversely, assume $1/p + 1/q + 1/r < 1$, following the proof of the corollary, we consider the asymptotic triangles $\triangle_{\alpha,\beta+\delta}, \triangle_{\pi-\gamma,\delta}$ as in the picture:



Where $\alpha = \pi/p, \beta = \pi/q, \gamma = \pi/r$, and $\delta$ is the angle that makes the right side vertical, and the difference of these two triangles gives us a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$.

(b) Since the area of such a triangle is $\pi(1 - (1/p + 1/q + 1/r))$, we maximize $1/p + 1/q + 1/r$ in order to minimize the area. To maximize, we consider $f(p) := 1/p + 1/q + 1/r$, and take $f'(p) = -1/p^2$, and this is always negative, so $f(p)$ is decreasing, hence the maximum occurs when $p$ is minimized. Since if $p = 1$, then $1/p = 1 \not< 1$, so we must have $p = 2$. Then, consider $g(q) := 1/2 + 1/q + 1/r$, and take $g'(q) = -1/q^2$, and likewise $g$ is maximized when $q$ is minimized. If $q = 2$, then $1/p + 1/q = 1 \not< 1$, so we must have $q = 3$. Considering $h(r) = 1/2 + 1/3 + 1/r$, the same argument yields that $r$ is the smallest integer such that $1/2 + 1/3 + 1/r < 1$, and therefore $1/r < 1 - 5/6 = 1/6 \implies r = 7$. So the hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$ for $p, q, r \in \mathbb{N}$ with smallest area is with $p = 2, q = 3, r = 7$.

(c) The idea is that we can reflect across any of the sides of the initial triangle, yielding an adjacent congruent triangle, and we may repeat this process infinitely many times, reflecting across different sides of each triangle, to generate infinitely many triangles in $\mathbb{H}^2$. Then, we need to show that there are no "gaps" or overlaps. In particular, given any single triangle generated, we show that each vertex is shared by triangles whose angles at the vertex add up to exactly $2\pi$. At the vertex with angle $\pi/2$, a reflection across one side gives a second triangle, whose angle at this vertex is also $\pi/2$. Reflect two more times along the other edge at this vertex, we get a total of four triangles whose angle sum at this vertex add up to precisely $2\pi$, thus there are no gaps or overlaps here. In the same way, for the vertex with angle $\pi/3$, we have 6 triangles with angle sum at this vertex adding up to $2\pi$, and for the vertex with angle $\pi/7$, we have 14 such triangles. As we apply these reflections indefinitely, these triangles cover all of $\mathbb{H}^2$, hence we have a tiling of $\mathbb{H}^2$ by triangles with angles $\pi/2, \pi/3, \pi/7$.

(d) In words, the hyperbolic isometries that preserve the tiling are the ones that send one of the triangles to another triangle in the tiling. From part (c), we've constructed the tiling using only reflections across the sides of the triangle; let's denote them $r_1, r_2, r_3$ for the reflections across the lines opposite the angles $\pi/2, \pi/3, \pi/7$ respectively. The hyperbolic isometries preserving the tiling are compositions of $r_1, r_2, r_3$, and since reflections across intersecting lines is a rotation about the point of intersection by twice the angle between them, we have the following relations:

$$r_1 r_2 \approx R_{2\pi/7}, r_1 r_3 \approx R_{2\pi/3}, r_2 r_3 \approx R_{\pi},$$

and hence

$$(r_1 r_2)^7 = (r_1 r_3)^3 = (r_2 r_3)^2 = \text{Id}.$$

Naturally, we also have $r_i^2 = \text{Id}$ for $i = 1, 2, 3$, and thus the group of isometries preserving the tiling is

$$\langle r_1, r_2, r_3 | r_i^2 = (r_1 r_2)^7 = (r_1 r_3)^3 = (r_2 r_3)^2 = \text{Id}\rangle \leq \text{Isom}(\mathbb{H}^2).$$