University of California Davis **Name (Print):** ____________________
Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Sample Final Examination **June 11 2026**
Time Limit: 2 Hours

This examination document contains 7 pages, including this cover page, and 6 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	18	
2	18	
3	18	
4	18	
5	18	
6	10	
Total:	100	

1. (18 points) (**Euclidean** $\mathbb{R}^2$) Consider the two points $P = (1, 2), Q = (-1, -5) \in \mathbb{R}^2$ in the Euclidean plane. Solve the following parts:

   (a) (5 points) Find the Euclidean distance between P and Q.

   (b) (5 points) Let $M = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and consider the unique line $L \subseteq \mathbb{R}^2$ equidistant to P and Q. Determine the image of Q under the isometry $r_M \circ r_L$.

   (c) (5 points) Find all the fixed points of the isometry $r_M \circ r_L$.

   (d) (3 points) Show that there is no line $N \subseteq \mathbb{R}^2$ such that $r_N \circ r_M \circ r_L = \text{id}$.

2. (18 points) (Γ**-Geometry for the cylinder**) Let $C = \mathbb{R}^2/\Gamma$ be the Euclidean cylinder, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the translation

$$t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

(a) (5 points) Draw the Γ-orbits ΓP and ΓQ in $\mathbb{R}^2$ of the two points

$$P = (-1, 4), Q = (3.25, -7.75) \in \mathbb{R}^2.$$

(b) (5 points) Compute the distance in C from ΓP to ΓQ.

(c) (5 points) Show that there are infinitely many lines in C through ΓP and ΓQ.

(d) (3 points) Consider the projections $\pi(L_1), \pi(L_2)$ to C of the two lines

$$L_1 = \{x = 0\} \subseteq \mathbb{R}^2, \quad L_2 = \{47x + y = 4\} \subseteq \mathbb{R}^2.$$

Explicitly find all the intersections points of $\pi(L_1)$ and $\pi(L_2)$ in C.

3. (18 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance.

(a) (5 points) Consider the points $P = (1, 0, 0) \in S^2$ and $Q = \frac{1}{\sqrt{3}}(1, -1, 1)$. Compute the spherical distance $d_{S^2}(P, Q)$ from P to Q.

(b) (5 points) Let $R_{P,\pi/2}, R_{Q,\pi/2} \in \mathrm{Isom}(S^2)$ be the rotations of angle $\pi/2$ centered at P and Q. Show that $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is a rotation and find its center and angle.

(c) (5 points) Determine whether $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is equal to $R_{Q,\pi/2} \circ R_{P,\pi/2}$.

(d) (3 points) Is there a spherical isometry $f \in \mathrm{Isom}(S^2)$ such that $f \circ (R_{P,\pi/2}, R_{Q,\pi/2})$ has no fixed points?

4. (18 points) (**Hyperbolic distances and lines in** $\mathbb{H}^2$) Let $P = i, Q = 3 + 4i \in \mathbb{H}^2$ be points in the hyperbolic upper-half plane $\mathbb{H}^2$. Solve the following parts:

   (a) (5 points) Show that $L = \{z \in \mathbb{H}^2 : |z - 4|^2 = 17\} \subseteq \mathbb{H}^2$ is the unique hyperbolic line through the points P and Q.

   (b) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$.

   (c) (5 points) Find the unique hyperbolic line M equidistant to P and Q.

   (d) (3 points) Let $r_L, r_M \in \text{Isom}(\mathbb{H}^2)$ be the hyperbolic inversions along L and M. Compute the image of P and Q under the composition $r_M \circ r_L$.

5. (18 points) (**Hyperbolic isometries in** $\mathbb{H}^2$) Consider the map $f, g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$

$$f(z) = \frac{\bar{z} + 9}{3\bar{z} - 5}.$$

(a) (5 points) Show that f is a hyperbolic isometry and it has no fixed points.

(b) (5 points) Let $g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be a hyperbolic isometry such that

$$g\left(-\frac{21}{17} + \frac{16}{17}i\right) = i, \quad g\left(-\frac{33}{61} + \frac{64}{61}i\right) = 2i, \quad g\left(-\frac{17}{13} + \frac{32}{13}i\right) = 1 + i.$$

Determine the image of the point $2 + 3i$ under the composition $f \circ g$.

(c) (5 points) Find a hyperbolic line $L \subseteq \mathbb{H}^2$ such that $f(L) = L$.

(d) (3 points) Show that there exists no $n \in \mathbb{N}$ such that $f^n = \mathrm{id}$.

6. (10 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

   (a) (2 points) Two different lines in the twisted cylinder cannot intersect only at two points.

   (1) True. (2) False.

   (b) (2 points) Any orientation-reversing hyperbolic isometry must have a fixed point.

   (1) True. (2) False.

   (c) (2 points) A spherical isometry is determined by the image of three distinct points.

   (1) True. (2) False.

   (d) (2 points) For any two points $P, Q \in S^2$, there exists a unique line between them.

   (1) True. (2) False.

   (e) (2 points) Any hyperbolic triangle with vertices in $\mathbb{H}^2$ must have area less than π.

   (1) True. (2) False.

   (f) (2 points) There exists a regular hyperbolic hexagon whose angles are all $\pi/2$.

   (1) True. (2) False.

   (g) (2 points) For any two distinct hyperbolic lines $L, M \subseteq \mathbb{H}^2$, there exists a hyperbolic isometry $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ with $f(L) = M$.

   (1) True. (2) False.

   (h) (2 points) If $L \subseteq \mathbb{R}^2$ is the line equidistant to $P, Q \in \mathbb{R}^2$, then any isometry fixing the points of L must send P to Q.

   (1) True. (2) False.

   (i) (2 points) If two hyperbolic lines L, M do not intersect, then there exists a unique hyperbolic line that is perpendicular to both L and M.

   (1) True. (2) False.