University of California Davis **Name (Print):** ____________________
Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Sample Final Examination **June 11 2026**
Time Limit: 2 Hours

This examination document contains 12 pages, including this cover page, and 6 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	18	
2	18	
3	18	
4	18	
5	18	
6	10	
Total:	100	

1. (18 points) (**Euclidean** $\mathbb{R}^2$) Consider the two points $P = (1, 2), Q = (-1, -5) \in \mathbb{R}^2$ in the Euclidean plane. Solve the following parts:

   (a) (5 points) Find the Euclidean distance between P and Q.
   $d(P, Q) = \sqrt{(-1-1)^2 + (-5-2)^2} = \sqrt{4 + 49} = \sqrt{53}$.

   (b) (5 points) Let $M = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and consider the unique line $L \subseteq \mathbb{R}^2$ equidistant to P and Q. Determine the image of Q under the isometry $r_M \circ r_L$.
   By a corollary from class, we know r_L swaps P and Q, so $r_L(Q) = P$. Since the reflection across the line $y = x$ precisely swaps the x and y-coordinates (this is a simple enough reflection where I do not think it's necessary to prove that it swaps the x and y-coordinates), we have that $r_M \circ r_L(Q) = r_M(P) = r_M(1, 2) = (2, 1)$.

   (c) (5 points) Find all the fixed points of the isometry $r_M \circ r_L$.
   The line L is the perpendicular bisector of $\overline{PQ}$, and $\overline{PQ}$ has slope $\frac{7}{2}$, hence L has slope $-\frac{2}{7}$. So M, L are not parallel, and by a Theorem from class, $r_M \circ r_L$ is a rotation with exactly one fixed point at the point of intersection of M, L. L passes through the midpoint of P, Q, which is $(\frac{1-1}{2}, \frac{2-5}{2}) = (0, -\frac{3}{2})$, so $L = \{(x, y) \in \mathbb{R}^2 : y + \frac{3}{2} = -\frac{2}{7}x\}$. M, L intersect when $x + \frac{3}{2} = -\frac{2}{7}x \implies \frac{9}{7}x = -\frac{3}{2} \implies x = -\frac{7}{6} = y$. So the fixed point of $r_M \circ r_L$ is $(-\frac{7}{6}, -\frac{7}{6})$.

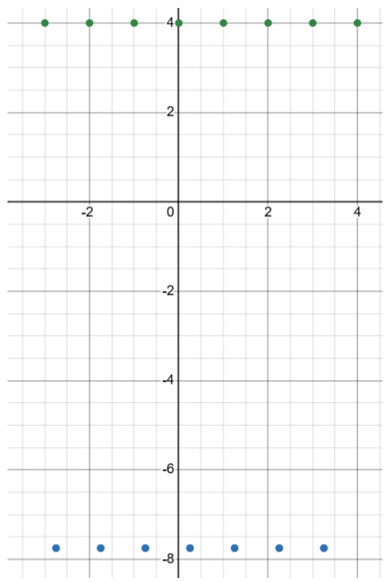
   (d) (3 points) Show that there is no line $N \subseteq \mathbb{R}^2$ such that $r_N \circ r_M \circ r_L = \text{id}$.
   We've deduced from part (c) that $r_M \circ r_L$ is a rotation, which is orientation-preserving. However, for any line N, r_N is orientation-reversing, and hence $r_N \circ r_M \circ r_L$ is orientation-reversing. But id is orientation-preserving, and it follows that $r_N \circ r_M \circ r_L \neq \text{id}$.

2. (18 points) (**Γ-Geometry for the cylinder**) Let $C = \mathbb{R}^2/\Gamma$ be the Euclidean cylinder, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the translation

$$t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

(a) (5 points) Draw the Γ-orbits ΓP and ΓQ in $\mathbb{R}^2$ of the two points

$$P = (-1, 4), Q = (3.25, -7.75) \in \mathbb{R}^2.$$



(b) (5 points) Compute the distance in C from ΓP to ΓQ.
We take the points $(0, 4) \in \Gamma P, (0.25, -7.75) \in \Gamma Q$, so $d_C(P, Q) = \sqrt{0.25^2 + 11.75^2}$.

(c) (5 points) Show that there are infinitely many lines in C through ΓP and ΓQ.
Consider the point $P \in \Gamma P$. For each $Q' \in \Gamma Q$, we get a line from P to Q', each with a different slope, hence they are all distinct when we quotient by Γ. Since there are infinitely many such Q', we conclude that there are infinitely many lines in C through ΓP and ΓQ.

(d) (3 points) Consider the projections $\pi(L_1), \pi(L_2)$ to C of the two lines

$$L_1 = \{x = 0\} \subseteq \mathbb{R}^2, \quad L_2 = \{47x + y = 4\} \subseteq \mathbb{R}^2.$$

Explicitly find all the intersections points of $\pi(L_1)$ and $\pi(L_2)$ in C.
Lifting to $\mathbb{R}^2$, we have that $\pi^{-1}(L_1)$ is identified with all lines $\{x = n\}$ for $n \in \mathbb{Z}$, so for $\pi(L_1), \pi(L_2)$ to intersect, we look for the points $(n, y) \in L_2$, which get mapped to $(0, y) \in \pi(L_1) \cap \pi(L_2)$. Rewriting, we have $L_2 = \{y = -47x + 4\}$, and thus the intersection points $(0, y)$ of $\pi(L_1) \cap \pi(L_2)$ are precisely $(0, -47n + 4)$ for all $n \in \mathbb{Z}$.

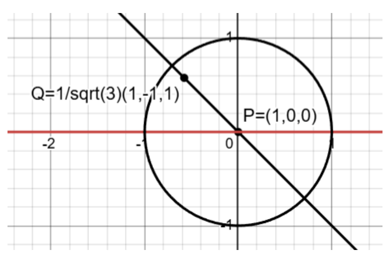
3. (18 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 = 1\},$$

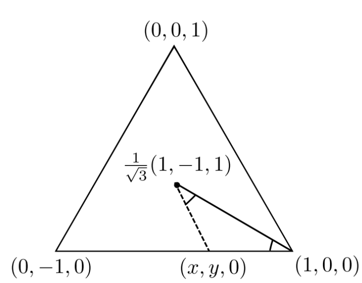
endowed with the spherical distance.

(a) (5 points) Consider the points $P = (1, 0, 0) \in S^2$ and $Q = \frac{1}{\sqrt{3}}(1, -1, 1)$. Compute the spherical distance $d_{S^2}(P, Q)$ from P to Q.
We compute the angle between $\langle 1, 0, 0\rangle$ and $\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$ using the formula $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \implies \theta = \cos^{-1}(\mathbf{u}\cdot\mathbf{v})$, so $d_{S^2}(P, Q) = \cos^{-1}(\frac{1}{\sqrt{3}})$.

(b) (5 points) Let $R_{P,\pi/2}, R_{Q,\pi/2} \in \mathrm{Isom}(S^2)$ be the rotations of angle $\pi/2$ centered at P and Q. Show that $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is a rotation and find its center and angle.
By a theorem from class, we know products of rotations on S^2 are also rotations, so $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is a rotation. Now, the hard part, using only things that we've learned: We know from class that rotations on S^2 centered at a point with angle θ are products of reflections across planes in $\mathbb{R}^2$ intersecting at the point and its antipodal, with angle of intersection $\theta/2$. Define G to be the unique plane through $O = (0, 0, 0), P, Q$. By choice of reflecting planes, we may have $R_{Q,\pi/2}, R_{P,\pi/2}$ be written as products of reflections, where one of the reflections is across G. So we write $R_{P,\pi/2} \circ R_{Q,\pi/2} = r_A \circ r_G \circ r_G \circ r_B = r_A \circ r_B$, where A, B are planes to be determined. The idea is that the intersection point $A \cap B \cap S^2$ will be the center of rotation, and the angle of rotation is twice the angle between A, B. The plane A is easier to find, by looking at the sphere from the x-axis (like a bird's eye view): We see that the line $z = 0$ forms an angle of $\frac{\pi}{4}$ with G, so A is the xy-plane. B is



a little more tricky. We know already that G, A form an angle of $\frac{\pi}{4}$, and we want B, G to form an angle of $\frac{\pi}{4}$ as well (be careful with the sign of the angle; we had the angle $\frac{\pi}{4}$ from G to A, and now we want the angle $\frac{\pi}{4}$ from B to G). Consider the following picture, viewing from the axis $y = -x, z = 0$:



By symmetry, if we were to have B, G form an angle of $\frac{\pi}{4}$, then we must have $d_{S^2}(P,(x,y,0)) = d_{S^2}(Q,(x,y,0))$, and $(x,y,0) \in S^2$. But we may also just compare the Euclidean distances between them, hence we have the equation

$$\begin{aligned}
&\sqrt{(x-1)^2+y^2} = \sqrt{\left(x-\frac{1}{\sqrt{3}}\right)^2 + \left(y+\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{3}} \\
&\iff (x-1)^2+y^2 = \left(x-\frac{1}{\sqrt{3}}\right)^2 + \left(y+\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{3} \\
&\iff x^2-2x+1+y^2 = x^2 - \frac{2}{\sqrt{3}}x + \frac{1}{3} + y^2 + \frac{2}{\sqrt{3}}y + \frac{1}{3} + \frac{1}{3} \\
&\iff \left(1-\frac{1}{\sqrt{3}}\right)x + \frac{1}{\sqrt{3}}y = 0 \\
&\implies y = -\left(1-\frac{1}{\sqrt{3}}\right)x \cdot \sqrt{3} = (1-\sqrt{3})x.
\end{aligned}$$

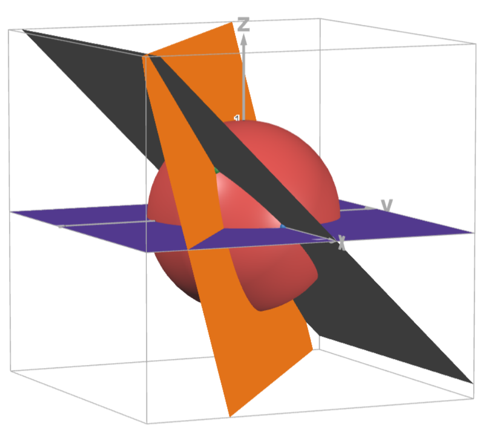
Then, we substitute this into the equation $x^2+y^2=1$ to get

$$x^2 + (1-\sqrt{3})^2x^2 = 1 \implies x^2 = \frac{1}{1+(1-\sqrt{3})^2} \implies x = \frac{1}{\sqrt{1+(1-\sqrt{3})^2}}$$

and

$$y = \frac{1-\sqrt{3}}{\sqrt{1+(1-\sqrt{3})^2}}.$$

So, B is the plane through $O, Q, \left(\frac{1}{\sqrt{1+(1-\sqrt{3})^2}}, \frac{1-\sqrt{3}}{\sqrt{1+(1-\sqrt{3})^2}}, 0\right)$. Using 21C methods, we can construct two vectors from these three points, compute their cross product to get a normal vector, and then using the 21C definition (or theorem?) that the equation of a plane through $(0,0,0)$ with normal vector $\langle a,b,c\rangle$ is $ax+by+cz=0$, we ultimately arrive at B being the plane $(\sqrt{3}-1)x + y + (2-\sqrt{3})z = 0$, or some scaling of this. Here's what the planes look like:

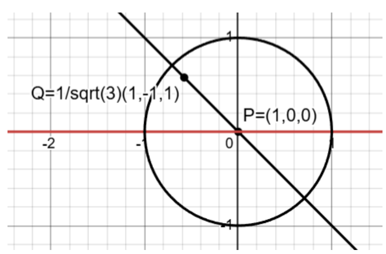


By the construction above, $\left(\frac{1}{\sqrt{1+(1-\sqrt{3})^2}}, \frac{1-\sqrt{3}}{\sqrt{1+(1-\sqrt{3})^2}}, 0\right)$ lies on both A and B, and therefore is fixed by $r_A \circ r_B$. That is, the center of rotation of $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is $\left(\frac{1}{\sqrt{1+(1-\sqrt{3})^2}}, \frac{1-\sqrt{3}}{\sqrt{1+(1-\sqrt{3})^2}}, 0\right)$. Finally, we compute the angle of rotation, which is twice the angle between A, B. From 21C, we know the angle between two planes is equal to the angle between their normal vectors. A has normal vector $\mathbf{n_1} = \langle 0, 0, 1\rangle$ and B has normal vector $\mathbf{n_2} = \langle \sqrt{3}-1, 1, 2-\sqrt{3}\rangle$. The angle between them is

$$\cos^{-1} \frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} = \cos^{-1} \frac{2-\sqrt{3}}{\sqrt{(\sqrt{3}-1)^2+1+(2-\sqrt{3})^2}} = \cos^{-1} \frac{2-\sqrt{3}}{\sqrt{12-6\sqrt{3}}} \approx 77.8^\circ$$

and thus the angle of rotation is $2\cos^{-1} \frac{2-\sqrt{3}}{\sqrt{12-6\sqrt{3}}} \approx 155.6^\circ$.

(c) (5 points) Determine whether $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is equal to $R_{Q,\pi/2} \circ R_{P,\pi/2}$.
We claim that $R_{P,\pi/2} \circ R_{Q,\pi/2} \neq R_{Q,\pi/2} \circ R_{P,\pi/2}$, and it suffices to show that $R_{Q,\pi/2} \circ R_{P,\pi/2}$ is a rotation with a different center. As with part (b), we write $R_{Q,\pi/2} \circ R_{P,\pi/2} = r_C \circ r_G \circ r_G \circ r_D = r_C \circ r_D$, where C, D are planes to be determined. From the same picture above, we see that the line $y = 0$ forms an angle of $\frac{\pi}{4}$ with G, so



D is the xz-plane. By symmetry, if we were to have G, C form an angle of $\frac{\pi}{4}$, then we must have $d_{S^2}(P, (x, 0, z)) = d_{S^2}(Q, (x, 0, z))$, and $(x, 0, z) \in S^2$, and a similar computation as above yields $z = (\sqrt{3}-1)x$, and substituting into $x^2 + z^2 = 1$ gives

$$x^2 + (\sqrt{3}-1)^2 x^2 = 1 \implies x = \frac{1}{\sqrt{1+(\sqrt{3}-1)^2}}, z = \frac{\sqrt{3}-1}{\sqrt{1+(\sqrt{3}-1)^2}}.$$

With the same reasoning as above, we deduce that $R_{Q,\pi/2} \circ R_{P,\pi/2}$ is a rotation with center $\left(\frac{1}{\sqrt{1+(\sqrt{3}-1)^2}}, 0, \frac{\sqrt{3}-1}{\sqrt{1+(\sqrt{3}-1)^2}}\right)$, and it follows that $R_{P,\pi/2} \circ R_{Q,\pi/2} \neq R_{Q,\pi/2} \circ R_{P,\pi/2}$.

(d) (3 points) Is there a spherical isometry $f \in \text{Isom}(S^2)$ such that $f \circ (R_{P,\pi/2}, R_{Q,\pi/2})$ has no fixed points?

Yes (I'm assuming the professor meant $f \circ (R_{P,\pi/2} \circ R_{Q,\pi/2})$). By above, we know that $R_{P,\pi/2} \circ R_{Q,\pi/2}$ is a rotation, which fixes only the center and its antipodal point. So we take f to be the reflection across the set of equidistant points to the center and its antipodal. Then clearly, $f \circ (R_{P,\pi/2} \circ R_{Q,\pi/2})$ does not fix the center and its antipodal. Points on the line of reflection remain on the line of reflection, but are translated by $R_{P,\pi/2} \circ R_{Q,\pi/2}$, then don't move by f. The remaining points get reflected to the opposite side of the line of reflection, hence we conclude that $f \circ (R_{P,\pi/2} \circ R_{Q,\pi/2})$ has no fixed points.

4. (18 points) (**Hyperbolic distances and lines in $\mathbb{H}^2$**) Let $P = i, Q = 3+4i \in \mathbb{H}^2$ be points in the hyperbolic upper-half plane $\mathbb{H}^2$. Solve the following parts:

   (a) (5 points) Show that $L = \{z \in \mathbb{H}^2 : |z-4|^2 = 17\} \subseteq \mathbb{H}^2$ is the unique hyperbolic line through the points P and Q.
   We construct the geodesic through P, Q. $\overline{PQ}$ has slope 1, so its perpendicular bisector has slope -1, and passes through $\frac{P+Q}{2} = \frac{3}{2} + \frac{5}{2}i$. So the perpendicular bisector has equation $y - \frac{5}{2} = -(x - \frac{3}{2}) = -x + \frac{3}{2}$. It intersects the x-axis at $y = 0$, so $-\frac{5}{2} = -x + \frac{3}{2} \implies x = 4$. Therefore, the center of the geodesic is $(4, 0)$. The radius is $d_{\mathbb{E}^2}((4,0),(0,1)) = \sqrt{17}$, and thus the geodesic through P, Q is $\{(x,y) \in \mathbb{H}^2 : (x-4)^2 + y^2 = 17\} = \{z \in \mathbb{H}^2 : |z-4|^2 = 17\} = L$.

   (b) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$.
   We have $y = \sqrt{17-(x-4)^2}$, $\frac{dy}{dx} = \frac{-(x-4)}{\sqrt{17-(x-4)^2}}$, and from P to Q, we have $0 \le x \le 3$, so

$$d_{\mathbb{H}^2}(P,Q) = \int_0^3 \frac{\sqrt{1 + \frac{(x-4)^2}{17-(x-4)^2}}}{\sqrt{17-(x-4)^2}} dx = \int_0^3 \frac{\sqrt{\frac{17}{17-(x-4)^2}}}{\sqrt{17-(x-4)^2}} dx$$
$$= \int_0^3 \frac{\sqrt{17}}{17-(x-4)^2} dx$$

   (c) (5 points) Find the unique hyperbolic line M equidistant to P and Q.
   First, we know from a theorem from class that reflection across M exchanges P, Q. In particular, M is perpendicular to the Euclidean line segment $\overline{PQ}$. By the construction of a circle, the radius from the center to any point on the circle forms an angle of $\pi/2$, and thus we may extend $\overline{PQ}$ to intersect the x-axis, from which we get that the center of M is $(-1, 0)$. Next, we find the radius. As M has center $(-1, 0)$, and must pass through some point on the segment $\overline{PQ}$, it follows that M must pass through the y-axis as well. So, consider a point $y_0 i \in \mathbb{H}^2$ equidistant to P, Q. We've seen from class that $d_{\mathbb{H}^2}(P, y_0 i) = \ln(y_0)$, and we set $d_{\mathbb{H}^2}(Q, y_0 i) = \ln(y_0)$. Let's derive $d_{\mathbb{H}^2}(Q, y_0 i)$: The perpendicular bisector has slope $-\frac{3}{4-y_0}$, and passes through the midpoint $\frac{3+(4+y_0)i}{2}$, from which we get the equation of the perpendicular bisector is $y - \frac{4+y_0}{2} = -\frac{3}{4-y_0}(x - \frac{3}{2})$. When $y = 0$, we get the center of this geodesic to be $x = \frac{(4+y_0)(4-y_0)}{6} + \frac{3}{2} = \frac{16-y_0^2+9}{2} = \frac{25-y_0^2}{6}$. The radius is the Euclidean distance $d_{\mathbb{E}}(\frac{25-y_0^2}{6}, 3+4i) = \sqrt{\left(\frac{25-y_0^2-18}{6}\right)^2 + 16} = \sqrt{\left(\frac{7-y_0^2}{6}\right)^2 + 16}$. Thus, this geodesic has equation $\left(x - \frac{25-y_0^2}{6}\right)^2 + y^2 = \left(\frac{7-y_0^2}{6}\right)^2 + 16$. We have the following simplification of the distance formula for a geodesic with center $(\alpha, 0)$ and radius r:

$$\int \frac{\sqrt{1+(dy/dx)^2}}{y}dx = \int \frac{\sqrt{1+\frac{(-(x-\alpha))^2}{r^2-(x-\alpha)^2}}}{\sqrt{r^2-(x-\alpha)^2}}dx = \int \frac{\sqrt{\frac{r^2}{r^2-(x-\alpha)^2}}}{\sqrt{r^2-(x-\alpha)^2}}dx$$
$$= \int \frac{r}{r^2-(x-\alpha)^2}dx = \int \frac{A}{r-(x-\alpha)} + \frac{B}{r+(x-\alpha)}dx$$
$$\implies r = Ar + A(x-\alpha) + Br - B(x-\alpha)$$
$$\implies Ar + Br = r, A(x-\alpha) - B(x-\alpha) = 0$$
$$\implies A + B = 1 \implies A = B = \frac{1}{2}$$
$$= \int \frac{\frac{1}{2}}{r-(x-\alpha)} + \frac{\frac{1}{2}}{r+(x-\alpha)}dx$$
$$= \frac{1}{2}\left(-\ln|r-(x-\alpha)| + \ln|r+(x-\alpha)|\right) = \frac{1}{2}\ln\left|\frac{r+(x-\alpha)}{r-(x-\alpha)}\right|,$$

So we get:

$$d_{\mathbb{H}^2}(Q, y_0 i) = \int_0^3 \frac{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}}{\left(\frac{7-y_0^2}{6}\right)^2+16-\left(x-\frac{25-y_0^2}{6}\right)^2}dx = \frac{1}{2}\ln\left|\frac{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}+\left(x-\frac{25-y_0^2}{6}\right)}{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}-\left(x-\frac{25-y_0^2}{6}\right)}\right|_0^3$$
$$= \frac{1}{2}\ln\left|\frac{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}+\left(\frac{y_0^2-7}{6}\right)}{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}-\left(\frac{y_0^2-7}{6}\right)}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}-\frac{25-y_0^2}{6}}{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}+\frac{25-y_0^2}{6}}\right|.$$

Setting this equal to $\ln(y_0)$, we can simplify this to

$$y_0^2 = \frac{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}+\left(\frac{y_0^2-7}{6}\right)}{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}-\left(\frac{y_0^2-7}{6}\right)} \cdot \frac{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}+\frac{25-y_0^2}{6}}{\sqrt{\left(\frac{7-y_0^2}{6}\right)^2+16}-\frac{25-y_0^2}{6}}.$$

As this gets a bit too messy for me to TeX it, we shall skip the details. But one may solve this and get $y_0^2 = 7$, hence $y_0 = \sqrt{7}$. Finally, we compute the radius of M to be the Euclidean distance from the center $(-1, 0)$ to the point $(0, \sqrt{7})$, which is $\sqrt{1+7} = \sqrt{8}$. Hence, $M = \{(x, y) \in \mathbb{H}^2 : (x+1)^2 + y^2 = 8\}$.

(d) (3 points) Let $r_L, r_M \in \mathrm{Isom}(\mathbb{H}^2)$ be the hyperbolic inversions along L and M. Compute the image of P and Q under the composition $r_M \circ r_L$.
By part (a), we know $P, Q \in L$, and therefore $r_L(P) = P, r_L(Q) = Q$. We also know by a theorem from class that r_M swaps P and Q, and it follows that $r_M \circ r_L(P) = r_M(P) = Q, r_M \circ r_L(Q) = r_M(Q) = P$.

5. (18 points) (**Hyperbolic isometries in $\mathbb{H}^2$**) Consider the map $f, g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$

$$f(z) = \frac{\bar{z}+9}{3\bar{z}-5}.$$

(a) (5 points) Show that f is a hyperbolic isometry and it has no fixed points.
By a theorem from class, we know that orientation-reversing hyperbolic isometries are of the form $\overline{f}(z) = \frac{-a\bar{z}+b}{-c\bar{z}+d}$, with $ad-bc=1$. So it suffices to write f in this form. We normalize f by dividing by $\sqrt{(-1)\cdot(-5)-(-3)\cdot 9} = \sqrt{5+27} = \sqrt{32}$. So $f(z) = \frac{\bar{z}+9}{3\bar{z}-5} = \frac{\frac{1}{\sqrt{32}}(\bar{z}+9)}{\frac{1}{\sqrt{32}}(3\bar{z}-5)} = \frac{-\frac{-1}{\sqrt{32}}\bar{z}+\frac{9}{\sqrt{32}}}{-\frac{-3}{\sqrt{32}}\bar{z}-\frac{5}{\sqrt{32}}}$, and we check: $\frac{-1}{\sqrt{32}}\cdot\frac{-5}{\sqrt{32}} - \frac{-3}{\sqrt{32}}\cdot\frac{9}{\sqrt{32}} = \frac{5+27}{32} = 1$, and thus f is a hyperbolic isometry. Next, we assume that $z_0 = \frac{\overline{z_0}+9}{3\overline{z_0}-5}$ for some $z_0 = x+yi \in \mathbb{H}^2$. Then, we have

$$\begin{aligned} x+9-yi &= (x+yi)(3x-5-3yi) \\ &= 3x^2-5x-3xyi+3xyi-5yi+3y^2 \\ &\implies 3x^2-6x-9+3y^2 = 0, -4y = 0. \end{aligned}$$

The latter equation implies $y=0$, in which case $z_0 \notin \mathbb{H}^2$. Therefore, f has no fixed points.

(b) (5 points) Let $g : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be a hyperbolic isometry such that

$$g\left(-\frac{21}{17}+\frac{16}{17}i\right) = i, \quad g\left(-\frac{33}{61}+\frac{64}{61}i\right) = 2i, \quad g\left(-\frac{17}{13}+\frac{32}{13}i\right) = 1+i.$$

Determine the image of the point $2+3i$ under the composition $f \circ g$.
We first do a cheeky observation:

$$\begin{aligned} f(i) &= \frac{-i+9}{-3i-5} = \frac{-i+9}{-5-3i}\cdot\frac{-5+3i}{-5+3i} = \frac{5i+3-45+27i}{34} = -\frac{21}{17}+\frac{16}{17}i \\ f(2i) &= \frac{-2i+9}{-6i-5} = \frac{-2i+9}{-5-6i}\cdot\frac{-5+6i}{-5+6i} = \frac{10i+12-45+54i}{25+36} = -\frac{33}{61}+\frac{64}{61}i \\ f(1+i) &= \frac{1-i+9}{3-3i-5} = \frac{10-i}{-2-3i}\cdot\frac{-2+3i}{-2+3i} = \frac{-20+30i+2i+3}{4+9} = -\frac{17}{13}+\frac{32}{13}i. \end{aligned}$$

By a theorem from class, hyperbolic isometries are uniquely determined by the images of three noncollinear points, thus it follows that $g = f^{-1}$. Therefore, $f \circ g(2+3i) = 2+3i$.

(c) (5 points) Find a hyperbolic line $L \subseteq \mathbb{H}^2$ such that $f(L) = L$.
We look for the two fixed points on $\partial\mathbb{H}^2$, continuing from the computations in part (a). We have $0 = 3x^2-6x-9 \implies 0 = x^2-2x-3 = (x-3)(x+1) \implies x = -1, 3$. So the hyperbolic line invariant under f has endpoints at $-1, 3$, and thus has center $\frac{-1+3}{2} = 1$, radius 2. Therefore, $L = \{(x,y) \in \mathbb{H}^2 : (x-1)^2+y^2 = 4\}$.

(d) (3 points) Show that there exists no $n \in \mathbb{N}$ such that $f^n = \text{id}$.

Since f is orientation-reversing, we know that any product of odd numbers of f will also be orientation-reversing, hence could not be id. So to show that there exists no $n \in \mathbb{N}$ such that $f^n = \text{id}$, it suffices to check for even powers of f. We compute f^2:

$$f^2(z) = \frac{\overline{\frac{\overline{z}+9}{3\overline{z}-5}} + 9}{3\overline{\frac{\overline{z}+9}{3\overline{z}-5}} - 5} = \frac{\frac{z+9}{3z-5} + \frac{27z-45}{3z-5}}{\frac{3z+27}{3z-5} + \frac{-15z+25}{3z-5}} = \frac{28z-36}{-12z+52} = \frac{7z-9}{-3z+13}$$
$$\longleftrightarrow \begin{bmatrix} 7 & -9 \\ -3 & 13 \end{bmatrix} \sim \begin{bmatrix} \frac{7}{8} & -\frac{9}{8} \\ -\frac{3}{8} & \frac{13}{8} \end{bmatrix}.$$

Observe that given any two matrices $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}, \begin{bmatrix} e & -f \\ -g & h \end{bmatrix}$ with $a, b, c, d, e, f, g > 0$, we have $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} e & -f \\ -g & h \end{bmatrix} = \begin{bmatrix} ae+bg & -af-bh \\ -ce-dg & cf+dh \end{bmatrix}$, with entries $ae+bg, cf+dh > 0; -af-bh, -ce-dg < 0$. In particular, the top right and bottom left entry will never be 0, so the product of such matrices is never the identity. Since f^2 is of this form, it follows that $f^{2k} = (f^2)^k \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for any $k \in \mathbb{N}$. We conclude that there exists no $n \in \mathbb{N}$ such that $f^n = \text{id}$.

6. (10 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) Two different lines in the twisted cylinder cannot intersect only at two points.

(1) True. (2) False.

(b) (2 points) Any orientation-reversing hyperbolic isometry must have a fixed point.

(1) True. (2) False.

(c) (2 points) A spherical isometry is determined by the image of three distinct points.

(1) True. (2) False.

(d) (2 points) For any two points $P, Q \in S^2$, there exists a unique line between them.

(1) True. (2) False.

(e) (2 points) Any hyperbolic triangle with vertices in $\mathbb{H}^2$ must have area less than π.

(1) True. (2) False.

(f) (2 points) There exists a regular hyperbolic hexagon whose angles are all $\pi/2$.

(1) True. (2) False.

(g) (2 points) For any two distinct hyperbolic lines $L, M \subseteq \mathbb{H}^2$, there exists a hyperbolic isometry $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ with $f(L) = M$.

(1) True. (2) False.

(h) (2 points) If $L \subseteq \mathbb{R}^2$ is the line equidistant to $P, Q \in \mathbb{R}^2$, then any isometry fixing the points of L must send P to Q.

(1) True. (2) False.

(i) (2 points) If two hyperbolic lines L, M do not intersect, then there exists a unique hyperbolic line that is perpendicular to both L and M.

(1) True. (2) False.