

Sample Midterm Examination
Time Limit: 50 Minutes

May 1 2026

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- (A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.
- (B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.
- (C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- (D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

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1. (20 points) (**Rotations in \mathbb{R}^2**) Consider the two points $P = (0, 0), Q = (1, 0) \in \mathbb{R}^2$ in the Euclidean plane. Solve the following parts:
- (a) (5 points) Let $R_{P, \pi/2}$ be a rotation of angle $\pi/2$ centered at P . Compute the image $R_{P, \pi/2}(3, 3)$ of the point $(3, 3) \in \mathbb{R}^2$ under the isometry $R_{P, \pi/2}$.
- (b) (5 points) Let $R_{Q, -\pi/2}$ be a rotation of angle $-\pi/2$ centered at Q . Compute the image $R_{Q, -\pi/2}(4, 5)$ of the point $(4, 5) \in \mathbb{R}^2$ under the isometry $R_{Q, -\pi/2}$.
- (c) (5 points) Let $(x, y) \in \mathbb{R}^2$ be any point. Where does $(x, y) \in \mathbb{R}^2$ get send under the composition $R_{Q, -\pi/2} \circ R_{P, \pi/2}$?
- (d) (5 points) Show that $R_{Q, -\pi/2} \circ R_{P, \pi/2} = t_{(1,1)}$.

2. (20 points) (**Γ -Geometry for the 2-Torus**) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

- (a) (5 points) Draw the Γ -orbits of the two points $P = (2, 3), Q = (0.5, -7.5) \in \mathbb{R}^2$.

- (b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which contains $P \in \mathbb{R}^2$.

- (c) (5 points) Consider $P = (2, 3), Q = (0.5, -7.5) \in \mathbb{R}^2/\Gamma$ as points in the 2-torus. Show that the line $\{(x, y) \in T^2 : x = y\} \subseteq T^2$ contains both P and Q .

- (d) (5 points) Find *all* lines $L \subseteq T^2$ such that $P, Q \in L$.

3. (20 points) (**Geometry in the Twisted Cylinder**) In this problem, *all* points and lines are considered in the twisted cylinder $M = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \circ \bar{r} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$. Solve the following parts:

(a) (5 points) Consider the points $P = (0,0), Q = (0.9,0.2), R = (5.9,-0.2) \in M$. Find the three distances $d(P,Q), d(P,R), d(Q,R) \in M$.

(b) (5 points) Find the intersection points between the line $\{(x,y) \in M : x = 0.5\} \subseteq M$ and the line $\{(x,y) \in M : x = -y\} \subseteq M$.

(c) (5 points) Find two lines $K, L \subseteq M$ such that $|L \cap K| = 2$.

(d) (5 points) Show that given two points $S, T \in M$ in the complement of the line $H = \{(x,y) \in M : y = 0\} \subseteq M$, there exists a continuous path $\gamma \subseteq M$ from S to T such that $|H \cap \gamma| = 0$.

4. (20 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance.

- (a) (5 points) Show that there exists a unique line $L \subseteq S^2$ containing the points $(1, 0, 0)$ and $(0, 1, 0)$.

- (b) (5 points) Prove that there are infinitely many lines in S^2 containing the points $(1, 0, 0)$ and $(-1, 0, 0)$.

- (c) (10 points) Let $R_1 : S^2 \rightarrow S^2$ be the rotation with center $(1, 0, 0)$ and angle $\pi/2$, and $R_2 : S^2 \rightarrow S^2$ be the rotation with center $(0, 0, 1)$ and angle $\pi/2$. Determine whether the isometry $R_1 \circ R_2$ is equal to $R_2 \circ R_1$.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.
- (a) (2 points) Two lines $K, L \subseteq T^2$ cannot intersect at more than one point.
(1) True. (2) False.
- (b) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be generated by translations. Then an isometry $g \in \Gamma$ cannot have fixed points.
(1) True. (2) False.
- (c) (2 points) The composition of an even number of reflections cannot be a reflection.
(1) True. (2) False.
- (d) (2 points) A glide reflection admits infinitely many fixed points.
(1) True. (2) False.
- (e) (2 points) For any glide reflection $\bar{r}_1 \in \text{Iso}(\mathbb{R}^2)$, there exists a glide reflection $\bar{r}_2 \in \text{Iso}(\mathbb{R}^2)$ such that $\bar{r}_2 \circ \bar{r}_1 = \text{Id}$.
(1) True. (2) False.
- (f) (2 points) An isometry $f : S^2 \rightarrow S^2$ is uniquely determined by the image of three points.
(1) True. (2) False.
- (g) (2 points) There are no isometries in the Möbius band except for the identity.
(1) True. (2) False.
- (h) (2 points) The product of two rotations in S^2 is necessarily a rotation.
(1) True. (2) False.
- (i) (2 points) Any isometry $f : S^2 \rightarrow S^2$ can be written as the product of at most three rotations.
(1) True. (2) False.