University of California Davis **Name (Print):** ____________________

Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Sample Midterm Examination II **May 1 2026**

Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Isometries in $\mathbb{R}^2$**) Consider the three points $P = (0,0), Q = (1,0), R = (0,1) \in \mathbb{R}^2$ in the Euclidean plane. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = (2,2)$, $f(Q) = (2,3)$ and $f(R) = (3,2)$.

   (a) (5 points) Find the images $f(-1,0)$ and $f(8,2)$ of the points $(-1,0)$ and $(8,2)$ under the isometry f.

   (b) (5 points) Prove that the isometry f is not a translation, i.e. there exists no vector $(\alpha,\beta) \in \mathbb{R}^2$ such that $f = t_{(\alpha,\beta)}$.

   (c) (5 points) Show that there exists no point $S \in \mathbb{R}^2$ such that $f(S) = S$.

   (d) (5 points) Find a set of *at most* three reflection $\{\overline{r}_{L_1}, \overline{r}_{L_2}, \overline{r}_{L_3}\} \in \mathrm{Iso}(\mathbb{R}^2)$ such that f is a composition of these reflections.

2. (20 points) (Γ**-Geometry for the Klein Bottle**) Let $K = \mathbb{R}^2/\Gamma$ be the Euclidean Klein Bottle, where $\Gamma = \langle t_{(0,1)}, \overline{r} \circ t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$.

   (a) (5 points) Draw the Γ-orbits of the following points:

   $$P = (0,0), Q = (0.5, 2), R = (1, -5), S = (3, -232) \in \mathbb{R}^2.$$

   (b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which is *not* a square.

   (c) (5 points) Consider the lines

   $$L = \{(x,y) \in K : x = 2y\}, \quad M = \{(x,y) \in K : x = 0\}.$$

   Find *all* the intersection points $L \cap M$.

   (d) (5 points) Consider the line $N = \{(x,y) \in K : x = \pi \cdot y\}$. Is the number of intersection points $M \cap N$ finite or infinite ?

3. (20 points) (**The Cylinder**) In this problem, *all* points and lines are considered in the cylinder $C = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$. Solve the following parts:

   (a) (5 points) Consider the points $P = (0.5, 0), Q = (0.3, 0.2), R = (5.9, -0.2) \in M$. Find an isometry $f : C \longrightarrow C$ such that

   $$f(P) = (0.7, 0), \quad f(Q) = (0.5, -0.2), \quad f(R) = (6.1, 0.2).$$

   (b) (5 points) Find infinitely many distinct lines $\{L_i\} \subseteq C$, $i \in \mathbb{N}$, which contain P, Q, i.e. $P, Q \in L_i$, for all $i \in \mathbb{N}$.

   (c) (5 points) Let $t_{(0,\pi)} : C \longrightarrow C$ be a vertical translation, and $H = \langle t_{(0,\pi)} \rangle$ the group of isometries of C it generates. Does the H-orbit of the point $R \in C$ have limit points in the cylinder C ? (Justify your answer.)

   (d) (5 points) Consider the group $A = \langle t_{(0,\sqrt{2})}, t_{(0,1)} \rangle$ as a subgroup of the group of isometries of C. Prove that the A-orbit of P inside the cylinder C has limit points.

4. (20 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance. Solve the following parts:

(a) (5 points) Compute the distance between $(1, 0, 0)$ and $(0, 0, 1)$.

(b) (5 points) Determine the set of points in S^2 whose distance to $(0, 1, 0)$ equals their distance to $(0, 0, 1)$.

(c) (10 points) Let $E = \{z = 0\} \subseteq S^2$ be the equator. Find an orientation-reversing isometry $f : S^2 \longrightarrow S^2$ such that $f(E) = E$ but f has no fixed points on the equator.

5. (20 points) For each of the five sentences below, circle the **unique** correct answer. You do *not* need to justify your answer.

   (a) (2 points) Let $(0,0),(0.5,0) \in C$ be two points in the cylinder. The set of points equidistant to $(0,0)$ and $(0.5,0)$ consists of exactly:

   (1) A line, (2) Empty (3) Two lines (4) Infinite Lines

   (b) (2 points) Two lines $L, M \subseteq T^2$ in the two torus must have:

   (1) Finitely Many Intersection Points (2) Infinitely Many Intersection Points

   (3) No Intersection Points. (4) None of the other answers.

   (c) (2 points) A non-trivial subgroup $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ must:

   (1) Contain a non-trivial translation, (2) Be generated by at most two elements,

   (3) Be fixed point free, (4) Contain a product of reflections.

   (d) (2 points) There exists a unique isometry which fixes

   (1) Three collinear points (2) Three non-collinear points

   (3) Four collinear points (4) The origin.

   (e) (2 points) Let $f : S^2 \longrightarrow S^2$ be an isometry. Then

   (1) f cannot have fixed points,

   (2) f is a product of at most three reflections,

   (3) f is a product of at most two reflections,

   (4) the fixed point set of f must be two antipodal points.