University of California Davis
Euclidean Geometry MAT 141

Name (Print): ____________________
Student ID (Print): ____________________

Sample Midterm Examination II

Time Limit: 50 Minutes

May 1 2026

This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Isometries in $\mathbb{R}^2$**) Consider the three points $P = (0,0), Q = (1,0), R = (0,1) \in \mathbb{R}^2$ in the Euclidean plane. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = (2,2)$, $f(Q) = (2,3)$ and $f(R) = (3,2)$.

   (a) (5 points) Find the images $f(-1,0)$ and $f(8,2)$ of the points $(-1,0)$ and $(8,2)$ under the isometry f.
   First, we claim that f is the glide reflection $t_{(2,2)}\overline{r}_{y=x}$ (the reflection across the line $y = x$ followed by $t_{(2,2)}$). Note that $\overline{r}_{y=x}(x,y) = (y,x)$ for any $(x,y) \in \mathbb{R}^2$, so $t_{(2,2)}\overline{r}_{y=x}$ does indeed send $P \mapsto (2,2), Q \mapsto (2,3), R \mapsto (3,2)$, and by uniqueness of isometries on three noncollinear points, it follows that $f = t_{(2,2)}\overline{r}_{y=x}$. Then, we compute $f(-1,0) = t_{(2,2)}(0,-1) = (2,1)$, and $f(8,2) = t_{(2,2)}(2,8) = (4,10)$.

   (b) (5 points) Prove that the isometry f is not a translation, i.e. there exists no vector $(\alpha,\beta) \in \mathbb{R}^2$ such that $f = t_{(\alpha,\beta)}$.
   We showed in part (a) that f is a glide reflection, which is orientation-reversing. Since translations are orientation-preserving, f cannot be a translation.

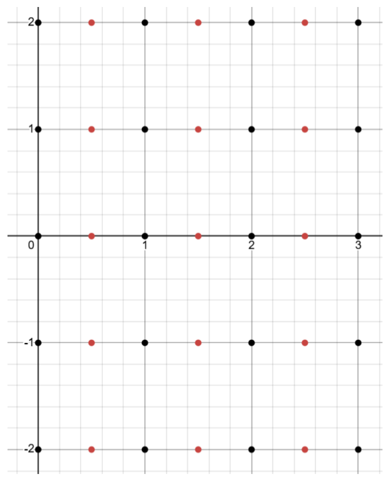
   (c) (5 points) Show that there exists no point $S \in \mathbb{R}^2$ such that $f(S) = S$.
   Assume for a contradiction that $f(x,y) = (x,y)$. We have $(x,y) = f(x,y) = t_{(2,2)}(y,x) = (y+2, x+2)$, which implies $y + 2 = x, x + 2 = y$, and it follows that $y + 4 = y$ and thus $4 = 0$, which is absurd. So there is no point $S \in \mathbb{R}^2$ such that $f(S) = S$.

   (d) (5 points) Find a set of *at most* three reflection $\{\overline{r}_{L_1}, \overline{r}_{L_2}, \overline{r}_{L_3}\} \in \text{Iso}(\mathbb{R}^2)$ such that f is a composition of these reflections.
   Let $L_1 = \{y = 2 - x\}, L_2 = \{y = -x\}, L_3 = \{y = x\}$. We have a Theorem from class that any translation is the product of two reflections, in particular a translation by d units in a given direction is the product of two reflections across lines perpendicular to the direction of d that are distance $\frac{d}{2}$ apart. So we note that $t_{(2,2)}$ is a translation by $2\sqrt{2}$ units along the line $y = x$, and L_1, L_2 are both perpendicular to $\{y = x\}$ and are distance $\sqrt{2}/2$ apart, so $t_{(2,2)} = \overline{r}_{L_1} \circ \overline{r}_{L_2}$, and it follows that $f = \overline{r}_{L_1} \circ \overline{r}_{L_2} \circ \overline{r}_{L_3}$. (One should check that this composition does indeed send P, Q, R to $f(P), f(Q), f(R)$, whether by explicit computations, or by drawing a picture of $\mathbb{R}^2$ and depicting how the points P, Q, R behave under the three reflections)

2. (20 points) (Γ**-Geometry for the Klein Bottle**) Let $K = \mathbb{R}^2/\Gamma$ be the Euclidean Klein Bottle, where $\Gamma = \langle t_{(0,1)}, \overline{r} \circ t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$.

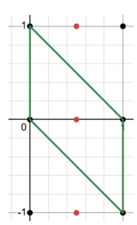
   (a) (5 points) Draw the Γ-orbits of the following points:

$$P = (0,0), Q = (0.5, 2), R = (1, -5), S = (3, -232) \in \mathbb{R}^2.$$



Note that $P \sim R \sim S$ since we can first apply a vertical translation by integer units to get R, S to lie on the x-axis, then apply the glide reflection to get to P. The black points are $\Gamma(P) = \Gamma(R) = \Gamma(S)$, and the red points are $\Gamma(Q)$.

   (b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which is *not* a square.



Starting from a unit square, we have that the two horizontal edges are identified, so we may "cut" the square into two pieces (in this picture, we're cutting along the line $y = 1 - x$), and glue the top piece to the bottom. Note that the resulting top and bottom edges of the parallelogram are identified via the translation $t_{(0,1)}$.

(c) (5 points) Consider the lines

$$L = \{(x, y) \in K : x = 2y\}, \quad M = \{(x, y) \in K : x = 0\}.$$

Find *all* the intersection points $L \cap M$.
Lifting to $\mathbb{R}^2$, we have $L = \{(x, \frac{x}{2}) : x \in \mathbb{R}\}, M = \{(0, y) : y \in \mathbb{R}\}$. In K, we have the two identifications: $(x, \frac{x}{2}) \sim (x, \frac{x}{2} + n), (x, \frac{x}{2}) \sim (x + n, (-1)^n(\frac{x}{2}))$ for any $n \in \mathbb{Z}$. In order for L, M to intersect, we must have $x + n = 0$, so $x \in \mathbb{Z}$. If x is even, then $\frac{x}{2} \in \mathbb{Z}$, so $(x, \frac{x}{2}) \sim (x, 0) \sim (0, 0)$, and if x is odd, then $\frac{x+1}{2} \in \mathbb{Z}$, so $(x, \frac{x}{2}) \sim (x - x, (-1)^x(\frac{x}{2})) = (0, -\frac{x}{2}) \sim (0, -\frac{x}{2} + \frac{x+1}{2}) = (0, \frac{1}{2})$. Thus, $L \cap M = \{(0, 0), (0, \frac{1}{2})\}$.

(d) (5 points) Consider the line $N = \{(x, y) \in K : x = \pi \cdot y\}$. Is the number of intersection points $M \cap N$ finite or infinite ?
We claim that $|M \cap N| = \infty$. Lifting to $\mathbb{R}^2$, we have that points in N are of the form $(x, \frac{x}{\pi}), x \in \mathbb{R}$. If M, N are to intersect, then we must have $x \in \mathbb{Z}$. In order for two intersection points $(0, \frac{x_1}{\pi}), (0, \frac{x_2}{\pi})$ (assuming $x_1 \neq x_2$) to be equivalent, we must have $\frac{x_1}{\pi} = \frac{x_2}{\pi} + n$ for some $n \in \mathbb{Z}$. Multiplying throughout by π, we have $x_1 = x_2 + n\pi \implies n\pi = x_1 - x_2 \in \mathbb{Z} \setminus \{0\}$, which is a contradiction because π is irrational. Since there are infinitely many distinct nonzero integers x_i, the number of intersection points $M \cap N$ must be infinite.

3. (20 points) (**The Cylinder**) In this problem, *all* points and lines are considered in the cylinder $C = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$. Solve the following parts:

   (a) (5 points) Consider the points $P = (0.5, 0), Q = (0.3, 0.2), R = (5.9, -0.2) \in M$. Find an isometry $f : C \longrightarrow C$ such that

   $$f(P) = (0.7, 0), \quad f(Q) = (0.5, -0.2), \quad f(R) = (6.1, 0.2).$$

   Let $f = t_{(0.2,0)} \circ \overline{r}$. We compute that $t_{(0.2,0)} \circ \overline{r}(P) = t_{(0.2,0)}(0.5, 0) = (0.7, 0), t_{(0.2,0)} \circ \overline{r}(Q) = t_{(0.2,0)}(0.3, -0.2) = (0.5, -0.2), t_{(0.2,0)} \circ \overline{r}(R) = t_{(0.2,0)}(5.9, 0.2) = (6.1, 0.2)$.

   (b) (5 points) Find infinitely many distinct lines $\{L_i\} \subseteq C$, $i \in \mathbb{N}$, which contain P, Q, i.e. $P, Q \in L_i$, for all $i \in \mathbb{N}$.
   Fixing the point P and defining $Q_i := (0.3 + i, 0.2) \in \Gamma(Q)$, we see that the the lines through P, Q_i have slope $\frac{0.2}{i-0.2}$. Such a line passes through $P = (0.5, 0)$, so we have the collection $\{L_i\} = \{y = \frac{0.2}{i-0.2}x - \frac{0.5 \cdot 0.2}{i-0.2}\}$ of infinitely many lines containing P, Q with different slopes, hence the L_i's are all distinct.

   (c) (5 points) Let $t_{(0,\pi)} : C \longrightarrow C$ be a vertical translation, and $H = \langle t_{(0,\pi)} \rangle$ the group of isometries of C it generates. Does the H-orbit of the point $R \in C$ have limit points in the cylinder C ? (Justify your answer.)
   No. The H-orbit of any point in the cylinder consists of points whose nearest points are distance π away, so any ϵ-neighborhood for $\epsilon < \frac{\pi}{2}$ of a point in the H-orbit will not contain another point in the H-orbit.

   (d) (5 points) Consider the group $A = \langle t_{(0,\sqrt{2})}, t_{(0,1)} \rangle$ as a subgroup of the group of isometries of C. Prove that the A-orbit of P inside the cylinder C has limit points.
   We prove that P is a limit point in $A(P)$. i.e. we want to construct a sequence of points $P_n \in A(P)$ so that $\lim_{n\to\infty} P_n = P$. Note that $A(P)$ consists of points of the form $(0.5, n\sqrt{2} + m)$ for $n, m \in \mathbb{Z}$ (in particular, the y-coordinate is a linear combination of $\sqrt{2}$ and 1). Observe that $|\sqrt{2} - 1| < 1$, so $\lim_{n\to\infty}(\sqrt{2} - 1)^n = 0$. By the binomial theorem, we have $(\sqrt{2} - 1)^n = \sum_{k=0}^{n} \binom{n}{k} \sqrt{2}^{n-k}$. We know $n - k \in \mathbb{Z}$, and if $n - k$ is even, then $\sqrt{2}^{n-k} \in \mathbb{Z}$, i.e., an integer multiple of 1. If $n - k$ is odd, then $\sqrt{2}^{n-k} = \sqrt{2}^{n-k-1}\sqrt{2}$ is an integer multiple of $\sqrt{2}$. By definition, $\binom{n}{k} \in \mathbb{Z}$, thus $(\sqrt{2} - 1)^n$ is a linear combination of $\sqrt{2}$ and 1. Now, define $P_n := (0.5, (\sqrt{2} - 1)^n) \in A(P)$. It follows that $\lim_{n\to\infty} P_n = P$.

4. (20 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance. Solve the following parts:

(a) (5 points) Compute the distance between $(1, 0, 0)$ and $(0, 0, 1)$.
Recall that the distance between two points on S^2 is the (shorter) arc length between the two points, which is equal to the angle between them. Note that the angle between $(1, 0, 0)$ and $(0, 0, 1)$ is $\frac{\pi}{2}$, hence $d_{S^2}((1, 0, 0), (0, 0, 1)) = \frac{\pi}{2}$.

(b) (5 points) Determine the set of points in S^2 whose distance to $(0, 1, 0)$ equals their distance to $(0, 0, 1)$.
The set of points in $\mathbb{R}^3$ equidistant to two points is a plane bisecting the segment between them. In particular, the segment between the two points is a normal vector to the plane, and the plane passes through the midpoint of the two points. The normal vector is $\langle 0, 1, -1\rangle$, and the midpoint is $(0, 0.5, 0.5)$. Then from 21C, we know that the equation of the plane through $(0, 0.5, 0.5)$ with normal vector $\langle 0, 1, -1\rangle$ is $(y - 0.5) - (z - 0.5) = 0 \iff z = y$. So the set of points in S^2 whose distance to $(0, 1, 0)$ equals their distance to $(0, 0, 1)$ is

$$\{z = y\} \cap S^2 = \{(x, y, y) \in \mathbb{R}^3 : x^2 + 2y^2 = 1\}.$$

(c) (10 points) Let $E = \{z = 0\} \subseteq S^2$ be the equator. Find an orientation-reversing isometry $f : S^2 \longrightarrow S^2$ such that $f(E) = E$ but f has no fixed points on the equator.
Let $R_{z,\theta}$ be the rotation about the z-axis with angle θ, and let $\overline{r}_{z=0}$ be the reflection across the xy-plane. Then define $f := R_{z,\theta} \circ \overline{r}_{z=0}$. By construction, f is a composition of isometries on S^2, hence f itself is an isometry on S^2. f is the product of an orientation-reversing isometry and an orientation-preserving isometry, hence f is orientation-reversing. Since $R_{z,\theta}, \overline{r}_{z=0}$ both map E to itself, we have that $f(E) = R_{z,\theta}(E) = E$. But $\overline{r}_{z=0}$ fixes all points on E while $R_{z,\theta}$ fixes no points on E whenever $\theta \not\equiv 0 \pmod{2\pi}$, it follows that f has no fixed points on E.

5. (20 points) For each of the five sentences below, circle the **unique** correct answer. You do *not* need to justify your answer.

   (a) (2 points) Let $(0, 0), (0.5, 0) \in C$ be two points in the cylinder. The set of points equidistant to $(0, 0)$ and $(0.5, 0)$ consists of exactly:

   (1) A line, (2) Empty (3) Two lines (4) Infinite Lines

   (b) (2 points) Two lines $L, M \subseteq T^2$ in the two torus must have:

   (1) Finitely Many Intersection Points (2) Infinitely Many Intersection Points

   (3) No Intersection Points. (4) None of the other answers.

   (c) (2 points) A non-trivial subgroup $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ must:

   (1) Contain a non-trivial translation, (2) Be generated by at most two elements,

   (3) Be fixed point free, (4) Contain a product of reflections.

   (d) (2 points) There exists a unique isometry which fixes

   (1) Three collinear points (2) Three non-collinear points

   (3) Four collinear points (4) The origin.

   (e) (2 points) Let $f : S^2 \longrightarrow S^2$ be an isometry. Then

   (1) f cannot have fixed points,

   (2) f is a product of at most three reflections,

   (3) f is a product of at most two reflections,

   (4) the fixed point set of f must be two antipodal points.