University of California Davis **Name (Print):** ____________________

Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Sample Midterm Examination III **May 1 2026**

Time Limit: 50 Minutes

This examination document contains 5 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
Total:	80	

1. (20 points) (**Properties of** $\Gamma \subseteq$ **Iso**$(\mathbb{R}^2)$) Solve the following parts:

   (a) (5 points) Let $L = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $M = \{(x, y) \in \mathbb{R}^2 : x = 6\}$. Find an element $g \in \Gamma = \langle \overline{r}_L, \overline{r}_M \rangle$ which has a *unique* fixed point.

   (b) (5 points) Draw the Γ-orbit of the point $(0, 0)$ where $\Gamma := \langle t_{(2,3)}, t_{(-1,0)} \rangle$.

   (c) (5 points) Give an instance of a line $L \in \mathbb{R}^2$ such that its image under the quotient $\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ is finite, where $\Gamma := \langle t_{(2,3)}, t_{(-1,0)} \rangle$ as in (b).

   (d) (5 points) Show that any non-trivial isometry in $\Gamma := \langle t_{(2,3)}, \overline{r} \circ t_{(1,0)} \rangle$ cannot have fixed points.

2. (20 points) (**Reflections in** $\mathbb{R}^2$) Consider the two lines $L_0 = \{y = 0\}, L_1 = \{x = y\} \subseteq \mathbb{R}^2$ and the two lines $M_0 = \{x = y + 1\}, M_1 = \{x = -y + 1\} \subseteq \mathbb{R}^2$.

   (a) (5 points) Show that the only fixed point of the isometry $\overline{r}_{L_1} \circ \overline{r}_{L_0}$ is $(0, 0)$.

   (b) (5 points) Prove that the isometry $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0}$ is a rotation.

   (c) (5 points) Show that there exist two lines $N_0, N_1 \subseteq \mathbb{R}^2$ such that

   $$\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0} = \overline{r}_{N_1} \circ \overline{r}_{N_0}.$$

   (d) (5 points) Find the image of a point $(x, y) \in \mathbb{R}^2$ under the isometry $\overline{r}_{M_0} \circ \overline{r}_{L_1}$.

3. (20 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance. Solve the following parts:

(a) (5 points) Consider the point $P = \frac{1}{\sqrt{3}}(1, 1, 1)$. Compute its spherical distance to each of the points $Q_1 = (1, 0, 0), Q_2 = (0, 1, 0)$ and $Q_3 = (0, 0, 1)$.

(b) (5 points) Let $L \subseteq S^2$ be the unique line containing P and Q_1, show that the point $\frac{1}{\sqrt{2}}(0, 1, 1)$ belongs to L.

(c) (5 points) Let E the unique line containing Q_1 and Q_2. Find the image of the unique line through Q_1 and Q_3 under the composition $\overline{r}_L \circ \overline{r}_E$.

(d) (5 points) Find the fixed points of the isometry f obtained by first applying $\overline{r}_L \circ \overline{r}_E$ and then applying the reflection $(x, y, z) \longrightarrow (-x, y, z)$.

4. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) For any pair of points $P, Q \in K$ in the Klein bottle, there are infinitely many distinct lines $L \subseteq K$ containing $P, Q \in K$.

(1) True. (2) False.

(b) (2 points) There exist rotations $R_{P,\theta}, R_{Q,\phi} \in \text{Iso}(\mathbb{R}^2)$ such that the composition $R_{P,\theta} \circ R_{Q,\phi}$ is *not* a rotation.

(1) True. (2) False.

(c) (2 points) Two lines $L, K \subseteq M$ in the twisted cylinder either intersect 0,1 or infinitely many times.

(1) True. (2) False.

(d) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be generated by a finite number of translations. Then there exists a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ of finite area.

(1) True. (2) False.

(e) (2 points) The quotient of the isometry $t_{(0,1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ gives a well-defined isometry in the twisted cylinder $\mathbb{R}^2/\langle t_{(1,0)} \circ \bar{r}\rangle$.

(1) True. (2) False.

(f) (2 points) There are no parallel lines in the cylinder.

(1) True. (2) False.

(g) (2 points) Every orientation-preserving isometry of the 2-sphere is a rotation.

(1) True. (2) False.

(h) (2 points) Every orientation-reserving isometry of the 2-sphere has a fixed point.

(1) True. (2) False.

(i) (2 points) There are parallel lines in the 2-sphere.

(1) True. (2) False.

(j) (2 points) The spherical distance between two points in the 2-sphere is the same as the Euclidean distance between these two points, as computed in $\mathbb{R}^3$.

(1) True. (2) False.