**University of California Davis** **Name (Print):** ____________________
**Euclidean Geometry MAT 141** **Student ID (Print):** ____________________

**Sample Midterm Examination III** **May 1 2026**
**Time Limit: 50 Minutes**

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This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

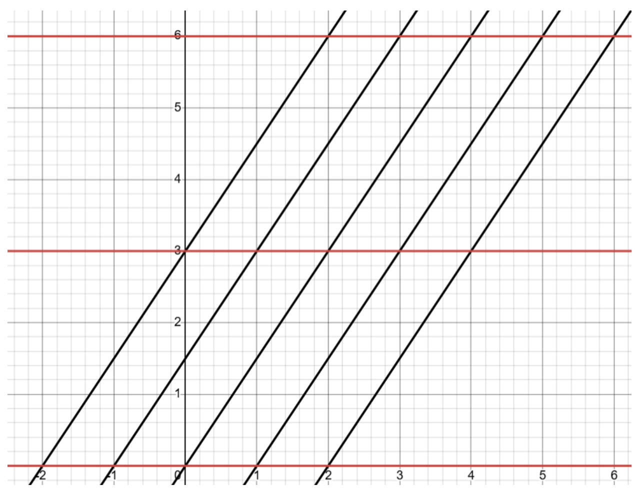
Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| Total: | 80 | |

1. (20 points) (**Properties of** $\Gamma \subseteq \mathbf{Iso}(\mathbb{R}^2)$) Solve the following parts:

   (a) (5 points) Let $L = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $M = \{(x, y) \in \mathbb{R}^2 : x = 6\}$. Find an element $g \in \Gamma = \langle \overline{r}_L, \overline{r}_M \rangle$ which has a *unique* fixed point.
   Note that $L, M$ intersect at a single point $(6, 6)$, so by a Theorem in class which shows the product of two reflections across intersecting lines is a rotation, $g := \overline{r}_L \overline{r}_M$ is a rotation, and therefore fixes a single point.

   (b) (5 points) Draw the $\Gamma$-orbit of the point $(0, 0)$ where $\Gamma := \langle t_{(2,3)}, t_{(-1,0)} \rangle$.
   $\Gamma(0, 0)$ consists of the intersection points of the black and red lines. The black lines have slope $\frac{3}{2}$, translated around by powers of $t_{(-1,0)}$, while the red lines have slope $\frac{0}{-1} = 0$, translated around by powers of $t_{(2,3)}$. (Side note: if we were to consider the quotient $\mathbb{R}^2/\Gamma$, then each of the parallelograms would be a fundamental domain). You may also just simply plot all the points instead of drawing the lines.



   (c) (5 points) Give an instance of a line $L \in \mathbb{R}^2$ such that its image under the quotient $\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ is finite, where $\Gamma := \langle t_{(2,3)}, t_{(-1,0)} \rangle$ as in $(b)$.
   The image of the line $L = \{y = 0\} \subseteq \mathbb{R}^2$ is a line segment $\{(x, 0) : 0 \leq x \leq 1\}$ with length 1.

(d) (5 points) Show that any non-trivial isometry in $\Gamma := \langle t_{(2,3)}, \overline{r} \circ t_{(1,0)} \rangle$ cannot have fixed points.

We want to show that if $f \in \Gamma$ is not the identity, then $f(x, y) \neq (x, y)$ for all $(x, y) \in \mathbb{R}^2$. By definition, $\Gamma$ consists of isometries which are compositions of $g := t_{(2,3)}$ and $h := \overline{r} \circ t_{(1,0)}$. First, we note that for any translation $t_{(a,b)}$, we have $\overline{r} \circ t_{(a,b)}(x, y) = \overline{r}(x + a, y + b) = (x + a, -y - b) = t_{(a,-b)} \circ \overline{r}(x, y)$. In particular, $t_{(a,b)}$ and $\overline{r}$ commute at the cost of a sign change to $t_{(a,-b)}$. Now, let $f \in \Gamma$ be nontrivial. By definition, $f$ is a word of the form $f = \prod_{i=1}^{k} g^{n_i} h^{m_i} = g^{n_k} h^{m_k} g^{n_{k-1}} h^{m_{k-1}} \cdots g^{n_1} h^{m_1}$. This is a product of translations and $\overline{r}$, which by the commutation formula $\overline{r} \circ t_{(a,b)} = t_{(a,-b)} \circ \overline{r}$, we may move all the $\overline{r}$ to the right, which gives us a single glide reflection if the total number of $h$ is odd, or a single translation if the total number of $h$ is even. Since $f \neq \mathrm{Id}$, we know that in the latter case where $f$ is a single translation, that it must not be the trivial translation, hence $f$ fixes no points. If $f$ is a single glide reflection, we need to show that the translation part is not trivial (otherwise, $f$ is a reflection, which has fixed points). To that end, it suffices to show the $x$-coordinate is not fixed. Since the number of $h$ is odd, there must be a translation $t_{(1,0)}$ somewhere. But since $h^2 = t_{(2,0)}$ and $g^n = t_{(2n,3n)}$, we must have $f = t_{(2m,3n)} t_{(1,0)} \overline{r}$ for some $m, n \in \mathbb{Z}$, but clearly $2m + 1 \neq 0$, hence the $x$-coordinate is not fixed. Thus, $f$ has no fixed points.

2. (20 points) (**Reflections in $\mathbb{R}^2$**) Consider the two lines $L_0 = \{y = 0\}, L_1 = \{x = y\} \subseteq \mathbb{R}^2$ and the two lines $M_0 = \{x = y + 1\}, M_1 = \{x = -y + 1\} \subseteq \mathbb{R}^2$.

   (a) (5 points) Show that the only fixed point of the isometry $\overline{r}_{L_1} \circ \overline{r}_{L_0}$ is $(0, 0)$.
   Recall a Theorem from class states that the product of two reflections across intersecting lines is a rotation about the intersection point. Since $L_0, L_1$ intersect only at the point $(0, 0)$, we know that the composition $\overline{r}_{L_1} \circ \overline{r}_{L_0}$ is a rotation about $(0, 0)$, hence the only fixed point is $(0, 0)$.

   (b) (5 points) Prove that the isometry $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0}$ is a rotation.
   $M_0, M_1$ are two lines intersecting at $(1, 0)$, so $\overline{r}_{M_1} \circ \overline{r}_{M_0}$ is a rotation about $(1, 0)$. In particular, the angle between $L_0, L_1$ is $\frac{\pi}{4}$, and the angle between $M_0, M_1$ is $\frac{\pi}{2}$, so $\overline{r}_{L_1} \circ \overline{r}_{L_0} = R_{\pi/2,(0,0)}$, and $\overline{r}_{M_1} \circ \overline{r}_{M_0} = R_{\pi,(1,0)}$. From the proof of the theorem (saw in week 2 discussion) for closure of the set of translations and rotations, we know that $R_{\theta,P} \circ R_{\phi,Q}$ for $P \neq Q$ is a translation if $\frac{\theta}{2} + \frac{\phi}{2} \equiv \pi \pmod{2\pi}$, and a rotation otherwise. Since $\frac{\pi}{4} + \frac{\pi}{2} \not\equiv \pi \pmod{2\pi}$, $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0} = R_{\pi,(1,0)} \circ R_{\pi/2,(0,0)}$ is a rotation.

   (c) (5 points) Show that there exist two lines $N_0, N_1 \subseteq \mathbb{R}^2$ such that

$$\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0} = \overline{r}_{N_1} \circ \overline{r}_{N_0}.$$

   By Theorem from class, we know that any rotation is a product of two reflections. Therefore, since $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0}$ is a rotation, there does exist lines $N_0, N_1 \subseteq \mathbb{R}^2$ such that $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0} = \overline{r}_{N_1} \circ \overline{r}_{N_0}$. (Note the question only asks us to show existence, so you don't have to find the explicit lines, though you are certainly welcome to.)

   (d) (5 points) Find the image of a point $(x, y) \in \mathbb{R}^2$ under the isometry $\overline{r}_{M_0} \circ \overline{r}_{L_1}$.
   Note that $L_1, M_0$ have the same slope, hence are parallel. So $\overline{r}_{M_0} \circ \overline{r}_{L_1}$ is a translation in the direction of the line $y = -x$ (since this line is perpendicular to the $L_1, M_0$) by twice the distance between $L_1, M_0$. The distance between $L_1, M_0$ is the distance between any perpendicular segment between two points on the lines, so we compute the distance between $(0, 0) \in L_1, (0.5, -0.5) \in M_0$, which is $\sqrt{2}/2$. So $\overline{r}_{M_0} \circ \overline{r}_{L_1} = t_{(a,-a)}$, where $\sqrt{a^2 + (-a)^2} = \sqrt{2a^2} = 2 \cdot \sqrt{2}/2$, thus $a = 1$. We conclude that $\overline{r}_{M_0} \circ \overline{r}_{L_1}(x, y) = t_{(1,-1)}(x, y) = (x + 1, y - 1)$. (Note that $L_1$ is above $M_0$, so the composition would be a translation towards the bottom right rather than the top left). You may also compute the formula explicitly by conjugation or some other method.

3. (20 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance. Solve the following parts:

(a) (5 points) Consider the point $P = \frac{1}{\sqrt{3}}(1, 1, 1)$. Compute its spherical distance to each of the points $Q_1 = (1, 0, 0), Q_2 = (0, 1, 0)$ and $Q_3 = (0, 0, 1)$.

We find the angles between $P$ and $Q_i$. By symmetry, these angles will all be the same.

$$\begin{aligned} d_{S^2}(P, Q_1) &= 2\sin^{-1}(\frac{1}{2}d_{\mathbb{R}^3}(P, Q_1)) \\ &= 2\sin^{-1}\left(\frac{1}{2}\sqrt{(1-\frac{1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2}\right) \\ &= 2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{(\sqrt{3}-1)^2}{3} + 2\cdot\frac{1}{3}}\right) \\ &= 2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3-2\sqrt{3}+1+2}{3}}\right), \end{aligned}$$

which is some ugly stuff that I would refuse to simplify much further on an exam. Alternatively, we may use 21C methods to find the angle between two vectors $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle, \langle 1, 0, 0\rangle$. Recall $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, so

$$\begin{aligned} \theta &= \cos^{-1}\frac{\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \cdot \langle 1, 0, 0\rangle}{|\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle||\langle 1, 0, 0\rangle|} \\ &= \cos^{-1}\frac{1}{\sqrt{3}}. \end{aligned}$$

So $d_{S^2}(P, Q_i) = \cos^{-1}\frac{1}{\sqrt{3}}$ for $i = 1, 2, 3$.

(b) (5 points) Let $L \subseteq S^2$ be the unique line containing $P$ and $Q_1$, show that the point $\frac{1}{\sqrt{2}}(0, 1, 1)$ belongs to $L$.

By symmetry, we have that $P, Q_1, (0, 0, 0)$ are all equidistant from $Q_2, Q_3$. That is, the line $L$ is the intersection of the plane equidistant from $Q_2, Q_3$ with $S^2$. So to show that $\frac{1}{\sqrt{2}}(0, 1, 1) \in L$, it suffices to show that it is equidistant from $Q_2 = (0, 1, 0), Q_3 = (0, 0, 1)$. But indeed by symmetry, we find that

$$d_{\mathbb{R}^3}\left(\frac{1}{\sqrt{2}}(0, 1, 1), (0, 1, 0)\right) = d_{\mathbb{R}^3}\left(\frac{1}{\sqrt{2}}(0, 1, 1), (0, 0, 1)\right),$$

and it follows that $\frac{1}{\sqrt{2}}(0, 1, 1) \in L$.

(c) (5 points) Let $E$ the unique line containing $Q_1$ and $Q_2$. Find the image of the unique line through $Q_1$ and $Q_3$ under the composition $\overline{r}_L \circ \overline{r}_E$.
First observe that $E$ is the equator, and the line $\mathcal{L}$ through $Q_1, Q_3$ is perpendicular to $E$, hence $\overline{r}_E(\mathcal{L}) = \mathcal{L}$. Then, we note that $L$ is the plane of equidistant points to $Q_3, Q_2$, hence $\overline{r}_L$ maps $Q_3$ to $Q_2$. But $Q_1 \in L$, so $\overline{r}_L(Q_1) = Q_1$, and it follows that $\overline{r}_L\overline{r}_E(\mathcal{L})$ is the unique line containing $Q_1, Q_2$, which is $E$.

(d) (5 points) Find the fixed points of the isometry $f$ obtained by first applying $\overline{r}_L \circ \overline{r}_E$ and then applying the reflection $(x, y, z) \longrightarrow (-x, y, z)$.
$\overline{r}_L\overline{r}_E$ is a product of reflections across planes intersecting at $Q_1, -Q_1$, so it is a nontrivial rotation about the $x$-axis. In particular, $\overline{r}_L\overline{r}_E(x, y, z) = (x, y', z')$ where $(y, z) \neq (y', z')$ if $(x, y, z) \neq Q_1, -Q_1$. Then applying the reflection $(x, y, z) \longrightarrow (-x, y, z)$ sends $(x, y', z') \longrightarrow (-x, y', z')$. Now, if $x = 0$, then $(x, y, z) \neq Q_1, -Q_1$, and it follows that $(x, y, z) \neq (-x, y', z')$. Otherwise, we have $x \neq 0$, and $x \neq -x$, so $(x, y, z) \neq (-x, y', z')$. Hence, $f$ has no fixed points.

4. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

   (a) (2 points) For any pair of points $P, Q \in K$ in the Klein bottle, there are infinitely many distinct lines $L \subseteq K$ containing $P, Q \in K$.

   (1) True. (2) False.

   (b) (2 points) There exist rotations $R_{P,\theta}, R_{Q,\phi} \in \text{Iso}(\mathbb{R}^2)$ such that the composition $R_{P,\theta} \circ R_{Q,\phi}$ is *not* a rotation.

   (1) True. (2) False.

   (c) (2 points) Two lines $L, K \subseteq M$ in the twisted cylinder either intersect 0,1 or infinitely many times.

   (1) True. (2) False.

   (d) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be generated by a finite number of translations. Then there exists a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ of finite area.

   (1) True. (2) False.

   (e) (2 points) The quotient of the isometry $t_{(0,1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ gives a well-defined isometry in the twisted cylinder $\mathbb{R}^2/\langle t_{(1,0)} \circ \bar{r} \rangle$.

   (1) True. (2) False.

   (f) (2 points) There are no parallel lines in the cylinder.

   (1) True. (2) False.

   (g) (2 points) Every orientation-preserving isometry of the 2-sphere is a rotation.

   (1) True. (2) False.

   (h) (2 points) Every orientation-reserving isometry of the 2-sphere has a fixed point.

   (1) True. (2) False.

   (i) (2 points) There are parallel lines in the 2-sphere.

   (1) True. (2) False.

   (j) (2 points) The spherical distance between two points in the 2-sphere is the same as the Euclidean distance between these two points, as computed in $\mathbb{R}^3$.

   (1) True. (2) False.