University of California Davis **Name (Print):** ______________________
Euclidean Geometry MAT 141 **Student ID (Print):** ______________________

Sample Midterm Examination **May 1 2026**
Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Rotations in $\mathbb{R}^2$**) Consider the two points $P = (0,0), Q = (1,0) \in \mathbb{R}^2$ in the Euclidean plane. Solve the following parts:

   (a) (5 points) Let $R_{P,\pi/2}$ be a rotation of angle $\pi/2$ centered at P. Compute the image $R_{P,\pi/2}(3,3)$ of the point $(3,3) \in \mathbb{R}^2$ under the isometry $R_{P,\pi/2}$.

   We have the formula $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, so $R_{P,\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, thus $R_{P,\frac{\pi}{2}}(3,3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = (-3,3)$.

   (b) (5 points) Let $R_{Q,-\pi/2}$ be a rotation of angle $-\pi/2$ centered at Q. Compute the image $R_{Q,-\pi/2}(4,5)$ of the point $(4,5) \in \mathbb{R}^2$ under the isometry $R_{Q,-\pi/2}$.

   By conjugation, we have $R_{Q,-\frac{\pi}{2}} = t_{(1,0)} R_{P,-\frac{\pi}{2}} t_{(-1,0)} = t_{(1,0)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t_{(-1,0)}$, so $R_{Q,-\frac{\pi}{2}}(4,5) = t_{(1,0)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = t_{(1,0)} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = (6,-3)$.

   (c) (5 points) Let $(x,y) \in \mathbb{R}^2$ be any point. Where does $(x,y) \in \mathbb{R}^2$ get send under the composition $R_{Q,-\pi/2} \circ R_{P,\pi/2}$?

$$\begin{aligned} R_{Q,-\frac{\pi}{2}} \circ R_{P,\frac{\pi}{2}}(x,y) &= t_{(1,0)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t_{(-1,0)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= t_{(1,0)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t_{(-1,0)} \begin{bmatrix} -y \\ x \end{bmatrix} \\ &= t_{(1,0)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -y-1 \\ x \end{bmatrix} \\ &= t_{(1,0)} \begin{bmatrix} x \\ y+1 \end{bmatrix} = (x+1, y+1). \end{aligned}$$

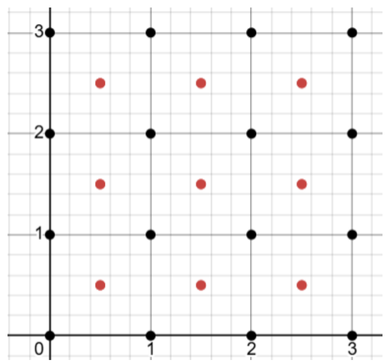
   (d) (5 points) Show that $R_{Q,-\pi/2} \circ R_{P,\pi/2} = t_{(1,1)}$.

   It is shown in part (c) that $R_{Q,-\frac{\pi}{2}} \circ R_{P,\frac{\pi}{2}} = t_{(1,1)}$.

2. (20 points) (**Γ-Geometry for the 2-Torus**) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

(a) (5 points) Draw the Γ-orbits of the two points $P = (2,3), Q = (0.5, -7.5) \in \mathbb{R}^2$.
The black points are $\Gamma(P)$ and the red points are $\Gamma(Q)$.



In particular $\Gamma(P) = \{(n,m) : n,m \in \mathbb{Z}\}$ and $\Gamma(Q) = \{(n+0.5, m+0.5) : n,m \in \mathbb{Z}\}$.

(b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which contains $P \in \mathbb{R}^2$.
A valid fundamental domain is the square $1 \leq x \leq 2, 2 \leq y \leq 3$.

(c) (5 points) Consider $P = (2,3), Q = (0.5, -7.5) \in \mathbb{R}^2/\Gamma$ as points in the 2-torus. Show that the line $\{(x,y) \in T^2 : x = y\} \subseteq T^2$ contains both P and Q.
The line $y = x$ goes through the points $(0,0), (0,5,0,5)$, and we know that on T^2, $(0,0) \sim (2,3) = P$ and $(0.5, 0.5) \sim (0.5, -7.5) = Q$, hence this line contains both P and Q.

(d) (5 points) Find *all* lines $L \subseteq T^2$ such that $P, Q \in L$.
We claim that the collection $\mathcal{L} := \{(x,y) \in T^2 : y = \frac{m+0.5}{n+0.5}x,\ \ m,n \in \mathbb{Z}\}$ form all the lines in T^2 containing P, Q. By construction, these lines all go through $(0,0) \sim P, (n+0.5, m+0.5) \sim Q$. So it remains to show that any line going through P, Q must be of this form. If a line $\widehat{L}$ goes through P, Q, then it must go through $(a,b), (c+0.5, d+0.5)$ for some $a,b,c,d \in \mathbb{Z}$, and we may apply the translation $t_{(-a,-b)}$ on everything, thus yielding the points $(0,0), (n+0.5, m+0.5)$, where $n = d-b, m = c-a$, and the line through these two points is $\widehat{L} = \{y = \frac{m+0.5}{n+0.5}x\}$, which is in $\mathcal{L}$.

3. (20 points) (**Geometry in the Twisted Cylinder**) In this problem, *all* points and lines are considered in the twisted cylinder $M = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \circ \overline{r} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$. Solve the following parts:

   (a) (5 points) Consider the points $P = (0, 0), Q = (0.9, 0.2), R = (5.9, -0.2) \in M$. Find the three distances $d(P, Q), d(P, R), d(Q, R) \in M$.
   $d(P, Q) = d(P, (-0.1, -0.2)) = \sqrt{0.1^2 + 0.2^2}$
   $d(P, R) = d(P, (-0.1, -0.2)) = \sqrt{0.1^2 + 0.2^2}$
   $d(Q, R) = 0$, since we find that $Q \sim R$ in M.

   (b) (5 points) Find the intersection points between the line $\{(x, y) \in M : x = 0.5\} \subseteq M$ and the line $\{(x, y) \in M : x = -y\} \subseteq M$.
   Define $L_1 = \{(0.5, y) : y \in \mathbb{R}\}, L_2 = \{(x, -x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$, and let $\pi : \mathbb{R}^2 \to M = \mathbb{R}^2/\Gamma$ be the natural projection. In M, we have the relation $(x, -x) \sim (x + n, (-1)^n(-x))$ for all $n \in \mathbb{Z}$. So for $\pi(L_1), \pi(L_2)$ to intersect in M, we must have $x = k+0.5$ for any $k \in \mathbb{Z}$. If k is even, say $k = 2j$ for some $j \in \mathbb{Z}$, then we have $(x, -x) = (2j+0.5, -2j-0.5) \sim (0.5, (-1)^{2j}(-2j-0.5)) = (0.5, -2j-0.5)$. If k is odd, say $k = 2i + 1$ for some $i \in \mathbb{Z}$, then we have $(x, -x) = (2i + 1.5, -2i - 1.5) \sim (0.5, 2i+1.5)$. So $\pi(L_1)\cap\pi(L_2) = \{(0.5, -2j-0.5) : j \in \mathbb{Z}\}\cup\{(0.5, 2i+1.5) : i \in \mathbb{Z}\}$. But we also observe that for any $2i + 1.5$, we may take $j = -(i + 1)$, so that $2i + 1.5 = 2(i + 1) - 0.5 = -2j - 0.5$, and thus the latter set is contained in the first, hence $\pi(L_1) \cap \pi(L_2) = \{(0.5, -2j - 0.5) : j \in \mathbb{Z}\}$.

   (c) (5 points) Find two lines $K, L \subseteq M$ such that $|L \cap K| = 2$.
   Consider $L = \{x = 0.5\}, K = \{y = 0.5\}$. Note that the horizontal line K traverses the segment $0 \le x \le 1, y = 0.5$, crossing L once. Then, since $(1, 0.5) \sim (0, -0.5)$, K traverses the segment $0 \le x \le 1, y = -0.5$, crossing L a second time. Since $(1, -0.5) \sim (0, 0.5)$, L_2 closes back up to its initial starting point. Thus, $|L\cap K| = 2$.

   (d) (5 points) Show that given two points $S, T \in M$ in the complement of the line $H = \{(x, y) \in M : y = 0\} \subseteq M$, there exists a continuous path $\gamma \subseteq M$ from S to T such that $|H \cap \gamma| = 0$.
   We may take S, T to be in the fundamental domain $0 \le x \le 1$. Write $S = (x_1, y_1), T = (x_2, y_2)$, with $y_1, y_2 \neq 0$. If $y_1y_2 > 0$ (i.e. either S, T are both above H, or both below H), then we may take γ to be the straight line through S, T (drawing such a line in the fundamental domain may not look continuous, but keep in mind that with the identification given by the quotient, these lines are indeed continuous in M). If $y_1y_2 < 0$, then note that $T = (x_2, y_2) \sim (x_2+1, -y_2)$, and $(x_2+1, -y_2), S$ are either both above H or both below H, so we may take γ to be the straight line through S and $(x_2 + 1, -y_2)$.

4. (20 points) (**Spherical geometry**) Consider the 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\},$$

endowed with the spherical distance.

(a) (5 points) Show that there exists a unique line $L \subseteq S^2$ containing the points $(1, 0, 0)$ and $(0, 1, 0)$.
By definition, a line in S^2 is the intersection of a plane through O in $\mathbb{R}^3$ with S^2. We know from 21C that there exists a unique plane through 3 noncollinear points in $\mathbb{R}^3$, and since $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ are noncollinear, this plane is unique, hence there exists a unique line through $(1, 0, 0), (0, 1, 0)$ in S^2.

(b) (5 points) Prove that there are infinitely many lines in S^2 containing the points $(1, 0, 0)$ and $(-1, 0, 0)$.
Since the three points $(0, 0, 0), (1, 0, 0), (-1, 0, 0)$ are collinear, there exists infinitely many planes containing the three points, hence there are infinitely many lines in S^2 containing $(1, 0, 0)$ and $(-1, 0, 0)$.

(c) (10 points) Let $R_1 : S^2 \longrightarrow S^2$ be the rotation with center $(1, 0, 0)$ and angle $\pi/2$, and $R_2 : S^2 \longrightarrow S^2$ be the rotation with center $(0, 0, 1)$ and angle $\pi/2$. Determine whether the isometry $R_1 \circ R_2$ is equal to $R_2 \circ R_1$.
We show that $R_1 \circ R_2 \neq R_2 \circ R_1$. Consider the point $(0, 1, 0)$, and observe that R_2 is just rotation about the origin in $\mathbb{R}^2$, with the z-coordinate fixed, so $(0, 1, 0)$ gets sent to $(-1, 0, 0)$. Then $(-1, 0, 0)$ is on the axis of revolution of R_1, so it is fixed by R_1, hence $R_1 \circ R_2(0, 1, 0) = R_1(-1, 0, 0) = (-1, 0, 0)$. Similarly, observe that R_1 is the rotation about the origin in the yz-plane, with the x-coordinate fixed, so $(0, 1, 0)$ gets sent to $(0, 0, 1)$. Then $(0, 0, 1)$ is on the axis of revolution of R_2, so it is fixed by R_2, hence $R_2 \circ R_1(0, 1, 0) = R_2(0, 0, 1) = (0, 0, 1)$.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) Two lines $K, L \subseteq T^2$ cannot intersect at more than one point.

(1) True. (2) False.

(b) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be generated by translations. Then an isometry $g \in \Gamma$ cannot have fixed points.

(1) True. (2) False.

(c) (2 points) The composition of an even number of reflections cannot be a reflection.

(1) True. (2) False.

(d) (2 points) A glide reflection admits infinitely many fixed points.

(1) True. (2) False.

(e) (2 points) For any glide reflection $\overline{r}_1 \in \text{Iso}(\mathbb{R}^2)$, there exists a glide reflection $\overline{r}_2 \in \text{Iso}(\mathbb{R}^2)$ such that $\overline{r}_2 \circ \overline{r}_1 = \text{Id}$.

(1) True. (2) False.

(f) (2 points) An isometry $f : S^2 \longrightarrow S^2$ is uniquely determined by the image of three points.

(1) True. (2) False.

(g) (2 points) There are no isometries in the Möbius band except for the identity.

(1) True. (2) False.

(h) (2 points) The product of two rotations in S^2 is necessarily a rotation.

(1) True. (2) False.

(i) (2 points) Any isometry $f : S^2 \longrightarrow S^2$ can be written as the product of at most three rotations.

(1) True. (2) False.