MAT 145: PROBLEM SET 1

DUE TO FRIDAY JAN 18

Abstract. This problem set corresponds to the beginning of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Tuesday Jan 8 and is due Friday Jan 18 at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the basic counts related to a set. In particular, we would like to become familiar with counts involving subsets, permutations and the binomial coefficients $\binom{n}{k}$ and the combinatorial arguments behind formulas and identities involving them.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Find the number of digits of 2^{432} and 3^{291}. Decide which of these two numbers is larger.

Problem 2. (1.5.5) We have 20 different presents that we want to distribute to 12 children. It is not required that every child get something; it could even happen that we give all the presents to the same child. In how many ways can we distribute the presents ?

Problem 3. (20 pts)

(a) Consider the set $X = \{2, 3, 4, 5, 6, 7, 8, 9\}$, which contains 8 elements. How many subsets of X have at least one prime number?

(The numbers 2, 3, 5 and 7 are the only prime numbers in X.)

(b) How many 10-character passwords can you create such that all the characters are numeric and distinct ?

(A character is numeric if it belongs to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.)

Problem 4. Let $n \in \mathbb{N}$ be a natural number, an ordered set of positive integers $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \ldots + \lambda_k = n$ is called a *composition* for $n \in \mathbb{N}$. These integers are not necessarily distinct. Show that the number of possible compositions for $n \in \mathbb{N}$ is 2^{n-1}.

Example: the number $n = 4$ has the following 8 compositions

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 3), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1).$$

Problem 5. (20 pts) Two parents go with their three kids to the theatre, and they have tickets for five consecutive seats.

(a) To avoid any chance of fraternal bickering, the parents decide that the kids cannot sit next to each other. How many different ways can the family sit together according to this rule if they seat along five consecutive seats ?

(b) Now, we allow kids to sit next to each. How many ways would there be if the only rule was that the two parents want to sit together?

Problem 6. (20 pts) Consider a 4×6 grid as depicted in Figure 1. Let us consider a path to be valid if it only move up or to the right along the grid. How many valid paths are there that go from the bottom left corner to the upper right corner ?

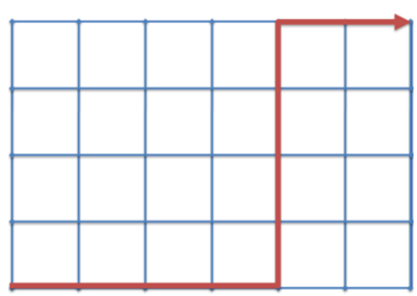

FIGURE 1. The 4×6 grid and a valid path drawn in red.

Problem 7. (1.8.5-1.8.6) (20 pts) Prove the following two formulas:

(i) For all $k, n \in \mathbb{N}$ with $k \leq n$,

$$\binom{n}{2} + \binom{n+1}{2} = n^2.$$

(ii) For all $k, n \in \mathbb{N}$ with $k \leq n$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

It is *strongly* recommended that you prove them by combinatorial means, rather than algebraic manipulation. However, a solution by either of the two methods will be given a full grade if correct.

Problem 8. Consider the tetrahedron depicted in Figure 2, it is a convex polyhedron composed of four triangular faces, six straight edges, and four vertex corners as drawn.

By definition, a *symmetry* of the tetrahedron is any spatial rotation along an axis (of any angle) or any spatial reflection (along any plane) which sends the tetrahedron to itself. A symmetry can thus exchange the vertices, edges and faces of the tetrahedron between them, but must in the end preserve the total shape of the tetrahedron itself as it sits in space.

Show that the number of symmetries of the tetrahedron is 24.

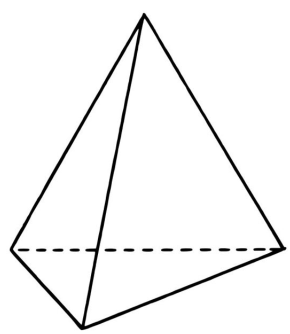

FIGURE 2. The Tetrahedron, one of the five Platonic Solids.