SOLUTIONS TO PROBLEM SET 1

MAT 145 COMBINATORICS

ABSTRACT. These are solutions corresponding to the Problem Set I of the Combinatorics Course in the Winter Quarter 2019. The Problem Set was posted online on Tuesday Jan 8 and is due Friday Jan 18 at the beginning of the class at 9:00am.

Information. These are the solutions for the Problem Set I corresponding to the Winter Quarter 2019 class of MAT 145 Combinatorics, with Prof. Casals and T.A. A. Aguirre. The Problems are written in black and the solutions in blue.

Problem 1. Find the number of digits of 2^{432} and 3^{291}. Decide which of these two numbers is larger.

Solution. The decimal expansion of 2^{432} has $\lfloor 432 \log_{10} 2 \rfloor + 1 = 131$ digits. The decimal expansion of 3^{291} has $\lfloor 291 \log_{10} 3 \rfloor + 1 = 139$ digits. Hence we have that $3^{291} > 2^{432}$.

Problem 2. (1.5.5) We have 20 different presents that we want to distribute to 12 children. It is not required that every child get something; it could even happen that we give all the presents to the same child. In how many ways can we distribute the presents ?

Solution. The key observation is that even though not every child might get something certainly every present is assigned to some child. For the first present we have 12 possible children to assign it to. For every succesive present (again since a given child might get as many as all the presents) we sill have 12 possible children we could give it to. The answer is thus 12^{20}.

Problem 3. (20 pts)

    (a) Consider the set $X = \{2, 3, 4, 5, 6, 7, 8, 9\}$, which contains 8 elements. How many subsets of X have at least one prime number?

        (The numbers 2, 3, 5 and 7 are the only prime numbers in X.)

    (b) How many 10-character passwords can you create such that all the characters are numeric and distinct ?

        (A character is numeric if it belongs to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.)

Solution.

(a) Let us denote by A the set of subsets that have at least one prime number. We wish to find $|A|$. Using the fact that $A \cap A^C = \emptyset$ and $A \cup A^C = X$ we have that $|A| = |\mathcal{P}(X)| - |A^C|$, where A^C denotes the set complement of A; in other words, the set of subsets of X that do not have any prime numbers. One can see then that A^C is the set of subsets of the subset of composite number in X which we denote by $B = \{4, 6, 8, 9\}$. That is $A^C = \mathcal{P}(B)$ and by the power set formula we have:

$$|A| = |\mathcal{P}(X)| - |A^C| = 2^8 - 2^4 = 240$$

(b) For the first character we have 10 options. For every succesive one we need to exclude a character (for all of them to be distinct). Therefore we have $10 \times 9 \times ... \times 2 \times 1 = 10!$ possible passwords.

Problem 4. Let $n \in \mathbb{N}$ be a natural number, an ordered set of positive integers $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \ldots + \lambda_k = n$ is called a *composition* for $n \in \mathbb{N}$. These integers are not necessarily distinct. Show that the number of possible compositions for $n \in \mathbb{N}$ is 2^{n-1}.

Example: the number $n = 4$ has the following 8 compositions
$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 3), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1).$$

Solution. Consider an array of n ones. A composition of n can be uniquely characterized by grouping the ones into blocks such that the fist block has λ_1 ones, the second block has λ_2 ones and so on. To "encode" such composition we can interlace $n - 1$ blank squares among the n 1's.

$$\left(\overbrace{1\square 1\square \ldots \square 1 \square 1}^{n} \right)$$

Each square can be either left empty or we can replace it by a plus sign. For instance for the composition $(2, 1, 1)$ of the number 4 we have $11 + 1 + 1$ (ignoring the squares). At each square we can either leave it blank or replace it by a plus sign. Clearly, there are 2^{n-1} ways to do this. Alternatively, we can induct on n using an argument very similar to the proof that $|\mathcal{P}(X)| = 2^n$ where $|X| = n$.

Problem 5. (20 pts) Two parents go with their three kids to the theatre, and they have tickets for five consecutive seats.

(a) To avoid any chance of fraternal bickering, the parents decide that the kids cannot sit next to each other. How many different ways can the family sit together according to this rule if they seat along five consecutive seats ?

(b) Now, we allow kids to sit next to each. How many ways would there be if the only rule was that the two parents want to sit together?

Solution.

(a) For no two kids to be next to each other, the only acceptable seating configurations are of the form KPKPK where K and P denote seats to be occupied by a kid and a parent respectively. Besides this there are no restrictions, hence in order to count the number of ways for them to sit, we simply count the number of ways we can permute the 3 kids (3!) and the 3 parents (2!) separately to obtain $3! \times 2! = 12$ acceptable seating configurations.

(b) If the parents are to sit together we can consider them to form a single block. In other words, up to permuting the two parents, arranging the seatings of the couple with the 3 kids is equivalent to the same task with 4 kids and no parents. This would give us 4! permutations. Accounting for the 2! permutations of the parents, our final answer is $4! \times 2! = 48$.

Problem 6. (20 pts) Consider a 4×6 grid as depicted in Figure 1. Let us consider a path to be valid if it only move up or to the right along the grid. How many valid paths are there that go from the bottom left corner to the upper right corner ?
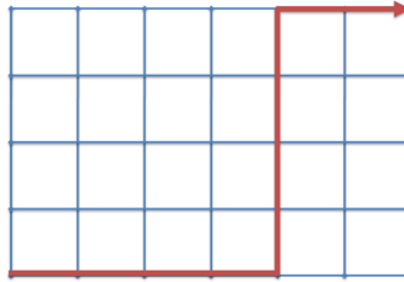

FIGURE 1. The 4×6 grid and a valid path drawn in red.

Solution. Since are only allowed to move up and right (i.e no backtracing ,circling etc.) every path will have exactly 10 steps upon arrival at the upper right corner. Thus every possible path is characterized by whether it went up or right at the k-th step (for each K up to 10). To count the number of all such possible paths, it is thus enough to count the ways in which we can choose the 4 up steps (since the rest will have to be right steps). This is given by the formula:

$$\binom{10}{4} = \frac{10!}{4!6!}$$

That is, cobinatorially speaking this problems reduces to the task of selecting k items out of n possible items. Note that we could have instead counted the number of ways to allocate the 6 right steps but we would obtain the same answer by the symmetry of the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \implies \binom{10}{4} = \binom{10}{6}$$

Problem 7. (1.8.5-1.8.6) (20 pts) Prove the following two formulas:

(i) For all $k, n \in \mathbb{N}$ with $k \leq n$,

$$\binom{n}{2} + \binom{n+1}{2} = n^2.$$

(ii) For all $k, n \in \mathbb{N}$ with $k \leq n$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

It is *strongly* recommended that you prove them by combinatorial means, rather than algebraic manipulation. However, a solution by either of the two methods will be given a full grade if correct.

Solution.

(i) First we verify algebraically that the identity holds:

$$\binom{n}{2} + \binom{n+1}{2} = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$$

For a combinatorial interpretation of this identity, let's consider the number of possible tuples (pairs of numbers) (a, b) with $1 \leq a, b \leq n$, which is clearly n^2. Now let's split the counting of such tuples into the three alternative situations $a < b$, $a > b$ and $a = b$. Notice that $\binom{n}{2}$ counts the amount of ways to pick two *distinct* elements from a set of n elements. Thus the $\binom{n}{2}$ part of the left hand side of our identity only takes care of the first set of tuples (those with $a < b$). For the remaining ones let us introduce an auxiliary element R to the set of the first n integers. By encoding the tuples of the form (m, m) by $\{m, R\}$ we can account for the tuples with $a > b$ as well as those with $a = b$ simply by choosing two elements from the enlarged set $\{1, 2, 3..., n\} \cup R$. Any time we choose $\{l, m\}$ where $l, m \in \{1, 2, 3..., n\}$ we are counting the other $\binom{n}{2}$ tuples of the form $a > b$. On the other hand whenever we choose $\{m, R\}$ where $m \in \{1, 2, 3..., n\}$ we are accounting for the n tuples of the form (m, m). Clearly the set $\{1, 2, 3..., n\} \cup R$ has cardinality $n+1$ and thus picking 2 distinct elements from it can be done in $\binom{n+1}{2}$ ways.

(ii) Algebraically, this identity rather immediate. For a combinatorial perspective, consider the task of selecting a cabinet of k people with a president out of n possible candidates. One can either:

(1) Select the president in n ways and the remianing $k-1$ members of the cabinet in $\binom{n-1}{k-1}$ ways for a total of $n\binom{n-1}{k-1}$ ways.

(2) Or select the k members of the cabinet first in $\binom{n}{k}$ ways and then the president among those in the cabinet in k ways for a total of $k\binom{n}{k}$ ways.

Problem 8. Consider the tetrahedron depicted in Figure 2, it is a convex polyhedron composed of four triangular faces, six straight edges, and four vertex corners as drawn.

By definition, a *symmetry* of the tetrahedron is any spatial rotation along an axis (of any angle) or any spatial reflection (along any plane) which sends the tetrahedron to itself. A symmetry can thus exchange the vertices, edges and faces of the tetrahedron between them, but must in the end preserve the total shape of the tetrahedron itself as it sits in space.

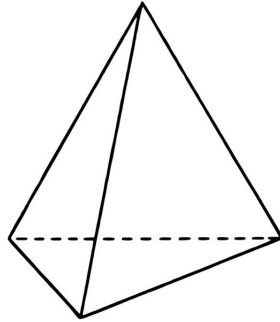Show that the number of symmetries of the tetrahedron is 24.

FIGURE 2. The Tetrahedron, one of the five Platonic Solids.

Solution. The idea is to note that any such symmetry preserves the shape of the tetrahedron and must therefore send vertices to vertices. In other words the symmetries permute the vertices. In the figure below we can see the two "operations" allowed. We can apply these to any face. If we fix our attention to the blue face, we can see that rotations induce the cyclic permutations of the vertices 1, 2 and 3 of the blue face. Likewise the reflection across the plane shown would permute vertices 1 and 3. In fact because we can rotate any of the 4 faces and permute any pair of vertices like this, by composing them we are able to obtain the 4! permutations of the label set of the vertices. Now this shows that there are at least 24 symmetries. It is also not hard to convince yourself that there are at most 24. In fact any symmetry would have to preserve the shape of the tetrahedron and thus the distances between vertices. Therefore modulo affine translations the symmetries would be precisely determined by which labels of the 1, 2 , 3 and 4 we assign to the coordinates we typically use for the tetrahedron namely, $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.
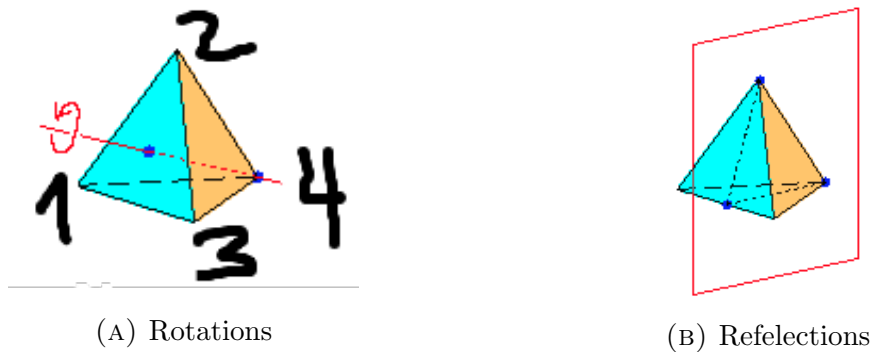

(A) Rotations

(B) Refelections

FIGURE 3. Symmetries of the regular tetrahedron.