MAT 145: PROBLEM SET 2

DUE TO FRIDAY JAN 25

ABSTRACT. This problem set corresponds to the second week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Tuesday Jan 15 and is due Friday Jan 25 at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the material covered during the second week of lectures. In particular, we would like to become familiar with proofs by induction, the use of the Inclusion-Exclusion Principle and the Pigeonhole Principle.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. By induction, prove the following two identities:

$$1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2}, \quad \forall n \in \mathbb{N},$$

and

$$1 + 4 + 7 + \ldots + (3(n-1) - 2) + (3n - 2) = \frac{n(3n-1)}{2}, \quad \forall n \in \mathbb{N}.$$

Problem 2. Show that $4^{n-1} > n^2$ for $n \geq 3$.

Hint: Use induction, and notice that the base case is $n = 3$.

Problem 3. (20 pts)

(a) (10 pts) Show that the following equality holds:

$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \binom{n+1}{2} + 2\binom{n+1}{3}, \quad \forall n \in \mathbb{N},$$

Hint: If you proceed by induction, you might want to use Theorem 1.8.2. If you search for a combinatorial proof, consider the set

$$X = \{(i, j, k) : 0 \le i, j < k \le n\}.$$

(b) (10 pts) Prove the following formula:

$$1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3 = \binom{n+1}{2}^2, \quad \forall n \in \mathbb{N},$$

Hint: As above, if you proceed by induction, you might want to use Theorem 1.8.2. If you want a combinatorial proof, you might want to consider the set

$$X = \{(i, j, k, l) : 0 \le i, j, k < l \le n\}.$$

Problem 4. (20 pts) Let $P(n)$ be an n-sided regular polygon in the plane. A *triangulation* of the polygon is a decomposition of the interior of the polygon into triangles, such that each triangle only intersects another triangle along one of its three sides. In Figure 1 I have depicted the different possible triangulations for the triangle $P(3)$, the square $P(4)$, the pentagon $P(5)$ and the hexagon $P(6)$.

Show that a triangulation of $P(n)$ must have exactly $(n-2)$ triangles.

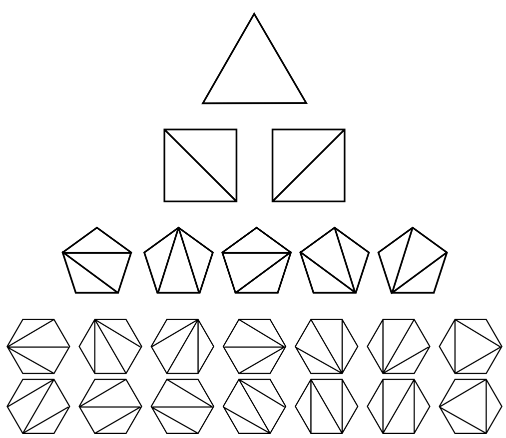

FIGURE 1. The triangle, the square with its two possible triangulations, the pentagon with its five possible triangulations and the hexagon with its fourteen possible triangulations.

Problem 5. (20 pts) Let $n \in \mathbb{N}$ be a natural number and let $X \subseteq \mathbb{N}$ be a subset with $n+1$ elements. Show that there exist two natural numbers $x, y \in X$ such that $x - y$ is divisible by n.

Problem 6. (20 pts) How many natural numbers $n \in \mathbb{N}$ between 1 and 100 are there which are *not* divisible by 5 nor divisible by 7 ?

Problem 7. (20 pts) Let $n \in \mathbb{N}$ be a natural number and X a finite set with n elements. Show that the number of permutations of X such that *no* element stays in the same position is

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

For instance, there are $6 = 3!$ permutations of 3 elements, but only 2 of them are permutations which fix *no* element. Similarly, there are $24 = 4!$ permutations of 4 elements, but only 9 which fix *no* element.

Hint: Use the Inclusion-Exclusion Principle, with the ith set being the set of permutations which fix the ith element of X.

Problem 8. Consider an infinite grid in the plane, and color every intersection with either red, green, or blue. Prove that for *any* possible choices of coloring, there always exists a rectangle in the plane such that all four of its vertices are the same color.

Figure 3 depicts a piece of such a grid with a rectangle with red vertices.

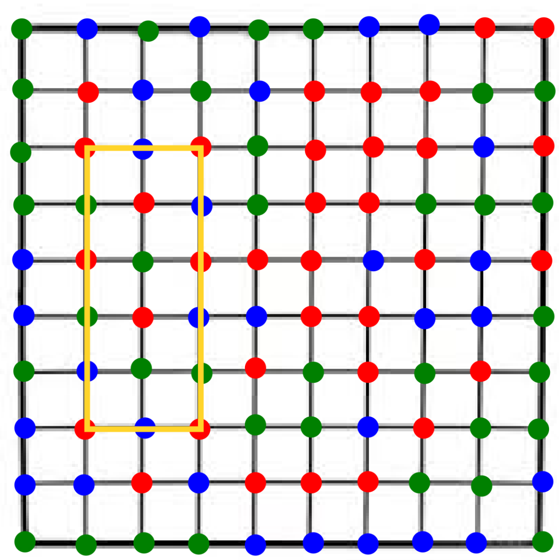

FIGURE 2. A piece of the grid colored with green, blue and red, and a yellow rectangle inside it with four red vertices.