SOLUTIONS TO PROBLEM SET 2

MAT 145 COMBINATORICS

ABSTRACT. These are solutions corresponding to the Problem Set 2 of the Combinatorics Course in the Winter Quarter 2019. The Problem Set was posted online on Tuesday Jan 15 and is due Friday Jan 25 at the beginning of the class at 9:00am.

Information. These are the solutions for the Problem Set 2 corresponding to the Winter Quarter 2019 class of MAT 145 Combinatorics, with Prof. Casals and T.A. A. Aguirre. The Problems are written in black and the solutions in blue.

Problem 1. By induction, prove the following two identities:

$$1+2+3+\ldots+(n-1)+n=\frac{n(n+1)}{2},\quad \forall n\in\mathbb{N},$$

and

$$1+4+7+\ldots+(3(n-1)-2)+(3n-2)=\frac{n(3n-1)}{2},\quad \forall n\in\mathbb{N}.$$

Solution. It is trivial to check the base case for $n=1$. Assume by inductive hypothesis that:

$$1+2+3+\ldots+(n-1)+n=\frac{n(n+1)}{2}$$

Adding $(n+1)$ on both sides:

$$1+2+3+\ldots+(n-1)+n=\frac{n(n+1)}{2}+(n+1)=(n+1)\left(\frac{n}{2}+1\right)=(n+1)(n+2)$$

Likewise for the other identity the base case is easy to verify. By inductive hypothesis assume that:

$$1+4+7+\ldots+(3(n-1)-2)+(3n-2)=\frac{n(3n-1)}{2}$$

Adding $3(n+1)-2$ on both sides:

$$1+4+\ldots+(3(n+1)-2)=\frac{n(3n-1)}{2}+(3(n+1)-2)=\frac{(3n^2-n)+(6n+2)}{2}=\frac{(n+1)(3(n+1)-1)}{2}$$

Problem 2. Show that $4^{n-1}>n^2$ for $n\geq 3$.

Hint: Use induction, and notice that the base case is $n=3$.

Solution. The base case is true since $4^2>3^2$. Assume by indictive hypothesis that $4^{n-1}>n^2$ for some n. Then multiplying by 4 on both sides of the inequality we have:

$$4^n > 4n^2 = (2n)^2 > (n+1)^2$$

Note that if $n \geq 3$ certainly $2n = n+n > n+1$ and the result follows because $f(x) = x^2$ is a strictly increasing function. $\square$

Problem 3. (20 pts)

(a) (10 pts) Show that the following equality holds:

$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \binom{n+1}{2} + 2\binom{n+1}{3}, \quad \forall n \in \mathbb{N},$$

Hint: If you proceed by induction, you might want to use Theorem 1.8.2. If you search for a combinatorial proof, consider the set

$$X = \{(i,j,k) : 0 \leq i,j < k \leq n\}.$$

(b) (10 pts) Prove the following formula:

$$1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3 = \binom{n+1}{2}^2, \quad \forall n \in \mathbb{N},$$

Hint: As above, if you proceed by induction, you might want to use Theorem 1.8.2. If you want a combinatorial proof, you might want to consider the set

$$X = \{(i,j,k,l) : 0 \leq i,j,k < l \leq n\}.$$

Solution. The inductive proofs are very similar to Problems 1 and 2, we shall thus present combinatorial proofs.

(a) For a combinatorial way of seeing this identity consider the set of triples of the form

$$\{(i,j,k) : 0 \leq i,j < k \leq n\}.$$

To count these we can condition on the value of k and then sum over $k = 0, 1, ...n-1, n$. In fact for a fixed k there are k^2 admissible values for i, j namely $0, 1, ...k-1$. Summing over k we obtain the left hand side. For the right hand side we divide the set of triples $\{(i,j,k) : 0 \leq i,j < k \leq n\}$ into the three possible cases $i = j$, $i < j$ and $i > j$. For $i = j$ we clearly have $\binom{n+1}{2}$ possible combinations. To see this note that we are choosing two *distinct* numbers from $\{0, 1, ...n\}$ the higher of which will be k while the lower one will be $i = j$. For each of the cases $i < j < k$ and $j < i < k$ we are instead choosing *three distinct* integers from $\{0, 1, ...n\}$ which shows the desired formula. $\square$

(b) Following Part (a), we can interpret the left hand side as the number of 4-tuples

$$X = \{(i, j, k, l) : 0 \leq i, j, k < l \leq n\}$$

by conditioning on the value of l and then summing over its possible values $l = 0, 1, ..., n-1, n$.

For the right hand side, the number $\binom{n+1}{2} \cdot \binom{n+1}{2}$ counts all pairs of tuples $\{(a, b), \quad (c, d)\}$ with $a < b$ and $c < d$. The set of such pairs of tuples is in bijection with $X = \{(i, j, k, l) : 0 \leq i, j, k < l \leq n\}$; in other words both sets have the same cardnality. Indeed, to construct the bijection let us split into the 3 cases $i < j$, $i > j$ and $i = k$. If $i < j$ we send (i, j, k, l) to $\{(i, j), (k, l)\}$ (sp that $i \neq j$ and $k \neq l$). If $i > j$ we likewise we send (i, j, k, l) to $\{(i, j), (k, l)\}$ (so that $i \neq j$ and $k \neq l$). Finally if $i = j$ we map (i, j, k, l) to $\{(k, l), (i, l)\}$ (so that $k \neq l$ and $i \neq l$). It might be a bit tricky to see how the inverse for this last part is unambiguous but its not hard to convince yourself that since $i = j$ there is only one possible value of j for the inverse of $\{(k, l), (i, l)\}$. □

Problem 4. (20 pts) Let $P(n)$ be an n-sided regular polygon in the plane. A *triangulation* of the polygon is a decomposition of the interior of the polygon into triangles, such that each triangle only intersects another triangle along one of its three sides. In Figure 1 I have depicted the different possible triangulations for the triangle $P(3)$, the square $P(4)$, the pentagon $P(5)$ and the hexagon $P(6)$.

Show that a triangulation of $P(n)$ must have exactly $(n-2)$ triangles.

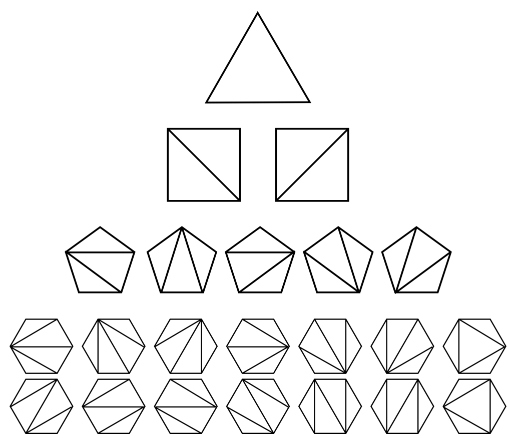

FIGURE 1. The triangle, the square with its two possible triangulations, the pentagon with its five possible triangulations and the hexagon with its fourteen possible triangulations.

Solution. It is easiest to proceed by *strong* induction in this exercise. This means that in the inductive step we not only assume that the statement holds for n but also for all $k \leq n$. The base case is trivial to check, a triangle has a unique triangulation namely itself. Now assume that every k-gon with $k \leq n$ has $(k-2)$ triangulations and let's consider an $(n+1)$-gon and draw a diagonal edge connecting two of its vertices. Suppose that on one side of the diagonal we have k of the original vertices from the $n+1$-gon (note $k \geq 2$ or else the diagonal we drew is one of the edges of the original

$n+1$-gon). Therefore together with the diagonal we have a $k+1$-gon with $k-1$ triangulations by strong inductive hypothesis. On the other side of the diagonal we have $n-k+1$ edges from the original $n+1$-gon which together with the diagonal form an $n-k+2$-gon. Since $k \geq 2$ (and thus $n-k+2 \leq n$) by strong inductive hypothesis we have $n-k$ triangulations on this side. Combining the triangulations on either side of the diagonal we have $(n-k)+(k-1) = n-1 = (n+1)-2$ triangulations in total which proves the desired result. □

Problem 5. (20 pts) Let $n \in \mathbb{N}$ be a natural number and let $X \subseteq \mathbb{N}$ be a subset with $n+1$ elements. Show that there exist two natural numbers $x, y \in X$ such that $x-y$ is divisible by n.

Solution. By the Pigeonhole Principle, if we have $n+1$ natural numbers, then at least two of them must belong to the same congruence class modulo n; in other words, the have the same reminder when you divide them by n. So we have at least one pair x, y such that $x = k_1 n + r$ and $y = k_2 n + r$ for some integers k_1, k_2. Therefore $x - y = (k_1 - k_2)n$ which shows the desired result. □

Problem 6. (20 pts) How many natural numbers $n \in \mathbb{N}$ between 1 and 100 are there which are *not* divisible by 5 nor divisible by 7 ?

Solution. Let A denote the set of natural numbers less than or equal to 100 that are divisible by 5 and B the set of natural numbers less than or equal to 100 that are divisible by 7. The set of of natural numbers less than or equal to 100 divisble by neither is thus $A^c \cap B^c = (A \cup B)^c$ where A^c and B^c are the respective complements in $\{1, 2, 3..., 99, 100\}$. Some set arithmetic and the Inclusion-Exclusion formula give us that:

$$|(A \cup B)^c| = 100 - |(A \cup B)| = 100 - (|A| + |B| - |A \cap B|) = 68$$

Clearly $|A| = 20$, $|B| = 14$ and $|A \cap B| = 2$ (multiples of 35 less than or equal to 100). □

Problem 7. (20 pts) Let $n \in \mathbb{N}$ be a natural number and X a finite set with n elements. Show that the number of permutations of X such that *no* element stays in the same position is

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

For instance, there are $6 = 3!$ permutations of 3 elements, but only 2 of them are permutations which fix *no* element. Similarly, there are $24 = 4!$ permutations of 4 elements, but only 9 which fix *no* element.

Hint: Use the Inclusion-Exclusion Principle, with the ith set being the set of permutations which fix the ith element of X.

Solution. Denote by D the set of permutations that fix no element and by S_n the whole set of permutations. Moreover, let A_i denote the set of permutations that fix the i-th element and $|A_i|$ be its cardinality. Now observe that D is the complement of the set of permutations that fix *at least one* element. The later set's cardinality is given by the Inclusion-Exclusion formula:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\leq i<j\leq n} |A_i \cap A_j| + \cdots \cdots + \sum_{1\leq i<j<k\leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

Here $|A_{i_1} \cap ... \cap A_{i_k}| = (n-k)!$ os the number of permutations that fix k labels (that is, the number of way to arrange the remaining $(n-k)$ labels). On the other hand clearly there are $\binom{n}{k}$ ways to choose those k labels on the first place. With this in mind and doing some simple set arithmetic:

$$|D| = |S_n| - \left|\bigcup_{i=1}^{n} A_i\right| = n! - \left(\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - ... + (-1)^{k-1}\binom{n}{n}0!\right)$$

$$= n! - \left(\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} ... + \frac{(-1)^{k-1} n!}{n!}\right) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

In the second to last equality we used $\binom{n}{k}(n-k)! = \frac{n!}{k!}$. □

Problem 8. Consider an infinite grid in the plane, and color every intersection with either red, green, or blue. Prove that for *any* possible choices of coloring, there always exists a rectangle in the plane such that all four of its vertices are the same color.

Figure 3 depicts a piece of such a grid with a rectangle with red vertices.

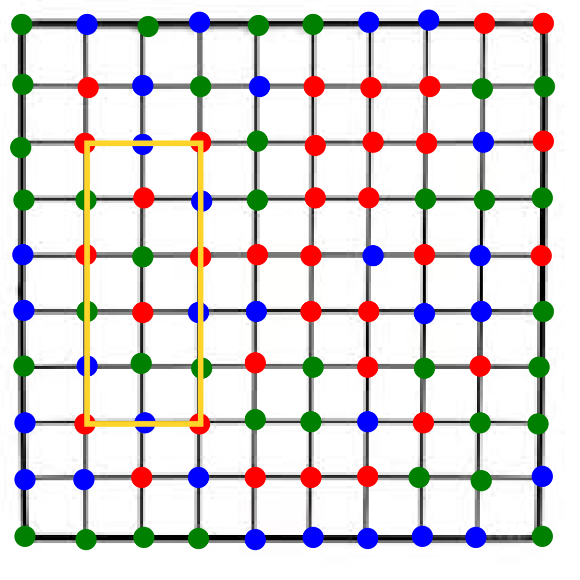

FIGURE 2. A piece of the grid colored with green, blue and red, and a yellow rectangle inside it with four red vertices.

Solution. In fact, a much smaller finite grid will suffice. Suppose we restrict our attention to for rows only and look at the infinitely many 4 element columns. Since we have 3 colors by the pigeonhole principle each column has at least one color repeat. Now let us look at the number of different arrangements (position in the column and color) in which this color repeat can happen. The color repeat is just a pair of points in the 4 point column with the same color. Therefore we have $\binom{4}{2} = 6$ color repeats per color for a total of 6×3 (3 colors). Thus if we have at least 19 columns by the pigeonhole principle there will be at least one parallel color repeat thereby forming a monochromatic rectangle. The situation is only better with an infinite gird. □