MAT 145: PROBLEM SET 3

DUE TO FRIDAY FEB 1

Abstract. This problem set corresponds to the third and fourth weeks of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Thursday Jan 24 and is due Friday Feb 1 at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the material covered during the third and fourth weeks of lectures. In particular, we would like to become familiar with identities in Pascal's Triangle, its Gaussian behavior and the basic aspects of combinatorial probability.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $n \in \mathbb{N}$, apply the Binomial Theorem to deduce the following two identities from Pascal's Triangle:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Problem 2. Find the coefficients of $x^{23}y^{45}$ in the expansion of $(x+y)^{68}$. Deduce that

$$\frac{d^{23}}{dx^{23}}|_{(x,y)=(1,0)}(x+y)^{68} = \frac{68!}{45!},$$

i.e. 23nd partial derivative with respect to x evaluated at $(x, y) = (1, 0)$ is $\frac{68!}{45!}$.

Problem 3. (20 pts)

(a) (10 pts) Show that the following identity in Pascal's Triangle holds:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \quad \forall n \in \mathbb{N},$$

(b) (10 pts) Prove the following formula, called the Hockey-Stick Identity:

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}, \quad \forall m, n \in \mathbb{N} \text{ with } m \leq n$$

Hint: If you want a combinatorial proof, consider the combinatorial problem of choosing a subset of $(n+1)$-elements from a set of $(n+m+1)$-elements, and separate by the cases given by the largest possible value in each of theses $(n+1)$-element subsets.

Problem 4. (20 pts) Consider a squared (167×167)-grid, and a path, traced across the grid, from the lower left corner to the upper right corner, which only moves to the right or upwards (these are called *staircase walks*).

Calculate the exact probability that a random path from the lower left corner to the upper right corner starts exactly by going to the right seven times, then two units up and then five units to the right again.

Problem 5. (20 pts) Consider three boxes and 12 balls, and exactly three of the balls are red. The first box fits any three balls, the second box fits any four balls and the third box will fit any five balls. Let us randomly put the balls into the boxes.

Show that the probability that all three red balls end up in the same box is $6.\widehat{81}\%$.

Hint: In this problem, we consider that there are three possible ways to fit a red ball inside the box of size three, four ways of fitting it inside a box of size four and five ways to fit one ball inside a box of size five. (Thus $\binom{5}{3}$ to fit three balls in that box of size five.) In addition, the three red balls are indistinguishable between them, and the nine blue balls are also indistinguishable between them.

Problem 6. (20 pts) By definition, a poker hand is a set of 5 cards from a standard French deck of 52 cards[1]. Solve the following two problems:

(a) A poker hand is said to be a *four of a kind* if it has four cards with the same value. For instance, four *sevens*, four *Queens* or four *Aces*. Compute the probability of drawing a four of a kind hand.

[1]We shall consider a standard French playing deck, which includes thirteen ranks in each of the four French suits. Thus, each suit will have 13 possible values.

(b) A poker hand is said to be a *full house* if it has three cards with the one value, and two cards with a second value. For instance, three *sevens* and two *Queens*, or three *Aces* and two *fives*. Compute the probability of drawing a full house.

Conclude why in the game of poker, a four of a kind beats a full house.

Problem 7. (20 pts) The following Problem comes from the game of Treize, featuring prominently in one of the earliest probability books, by P.R. de Montmort[2]. The game can be described as follows.

Consider the 13 cards of a given suit, so there are 13 different values, and draw them face down to the table. Each player calls out a number between 1 and 13 and turns one card face up. A player loses if the called value coincides with the card value. If no player loses by the time the last card is turned face up, then the dealer loses.

(a) Compute the probability that the dealer loses.

(b) Suppose that we played with 1982 distinct cards with values from 1 to 1982. Estimate the probability that the dealer loses.

Problem 8. Consider a squared $(n \times n)$-grid, and staircase walk that lies below, but is allowed to touch, the diagonal Δ formed by the points

$$\Delta = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y, 0 \leq x, y \leq n\}.$$

By definition, a *Dyck path* will be any such staircase walk below the diagonal. Find the probability that a staircase walk is a Dyck path.

Hint: This is actually of the same level of difficulty as the above problems.

[2]P. R. de Montmort, Essay d'Analyse sur des Jeux de Hazard, 2d ed. (Paris: Quillau, 1713), with the First Edition actually dating back to 1708. This problem is the First of the Exercises in Part 4 of this book, it reads "Déterminer generalement quel est à ce jeu l'avantage de celui qui tient les cartes.", i.e. "Find the expected value to the dealer when playing Treize.".