MAT 145: PROBLEM SET 3

DUE TO FRIDAY FEB 1

Abstract. These are solutions corresponding to the Problem Set 3 of the Combinatorics Course in the Winter Quarter 2019. The Problem Set was posted online on Thursday Jan 25 and is due Friday Jan 25 at the beginning of the class at 9:00am.

Information. These are the solutions for the Problem Set 3 corresponding to the Winter Quarter 2019 class of MAT 145 Combinatorics, with Prof. Casals and T.A. A. Aguirre. The Problems are written in black and the solutions in blue.

Problem 1. Let $n \in \mathbb{N}$, apply the Binomial Theorem to deduce the following two identities from Pascal's Triangle:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Solution. By the Binomial Theorem we have:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

Problem 2. Find the coefficients of $x^{23}y^{45}$ in the expansion of $(x+y)^{68}$. Deduce that

$$\frac{d^{23}}{dx^{23}}|_{(x,y)=(1,0)}(x+y)^{68} = \frac{68!}{45!},$$

i.e. 23nd partial derivative with respect to x evaluated at $(x,y) = (1,0)$ is $\frac{68!}{45!}$.

Solution. By the binomial theorem applied to $(x+y)^{68}$ the coefficient of $x^{23}y^{45}$ is $\binom{68}{23}$. Now, differentiating with respect to x 23 times we obtain:

$$\sum_{j=0}^{45} (23+j)! \binom{68}{23+j} x^j y^{45-j}$$

Notice that all the terms where the exponent of x was less than 23 are gone. Setting $x = 0$ and $y = 1$ all terms with $j > 0$ vanish and we are left with $23!\binom{68}{23} = \frac{68!}{45!}$. $\square$

Problem 3. (20 pts)

(a) (10 pts) Show that the following identity in Pascal's Triangle holds:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \quad \forall n \in \mathbb{N},$$

(b) (10 pts) Prove the following formula, called the Hockey-Stick Identity:

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}, \quad \forall m, n \in \mathbb{N} \text{ with } m \leq n$$

Hint: If you want a combinatorial proof, consider the combinatorial problem of choosing a subset of $(m+1)$*-elements from a set of* $(n+m+1)$*-elements.*

Solution. Part (a) The right hand side is counting the number of ways to choose n elements out of $2n$. As for the right hand side recall the identity $\binom{n}{k} = \binom{n}{n-k}$. Therefore we can rewrite it as:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

Now the k-th term of this sum can be interpreted as the number of ways of choosing k elements from the first half of a set of $2n$ elements and $n-k$ from the second half. Thus in total we are choosing n elements out of $2n$ and summing over all the possible k's we get precisely the number of ways to choose n elements out of $2n$. $\square$

Part (b) For the sake of the clarity we think in terms of sets of first n integers but the combinatorial reasoning extends to arbitrary sets. Clearly the right hand side is counting the number of ways to choose $n+1$ integers out of the first $n+m+1$. On the other hand the left hand side can be rewritten as:

$$\sum_{k=0}^{M} \binom{n+k}{n}$$

The k-th term of this element can be interpreted as the number of ways we can choose $n+1$ integers out of $\{1, 2, ...n+m+1\}$ where the first n of those are chosen from $\{1, 2, ...n+k\}$. Because the first n integers must appear between n and $n+m$ the k index runs from 0 to m and therefore the above sum counts precisely the number of ways to choose $n+1$ integers out of the first $n+m+1$. $\square$

Problem 4. (20 pts) Consider a squared (167×167)-grid, and a path, traced across the grid, from the lower left corner to the upper right corner, which only moves to the right or upwards (these are called *staircase walks*).

Estimate the probability that a random path from the lower left corner to the upper right corner starts by going to the right seven times, then two units up and then five units to the right again.

Hint: First find an exact formula for that probability, and then estimate it using our Gaussian approximation of Pascal's Triangle (Chapter 3).

Solution. Let us endow the grid with first quadrant coordinates. After the given steps we are at $(12, 2)$ and we still have 165 up moves and 155 right moves left for a total of 320. We can do so in $\binom{320}{155}$. The probability of a path starting this way is therefore the fraction of the number of these paths and all possible paths namely:

$$\frac{\binom{320}{155}}{\binom{2\times 167}{167}}.$$

Problem 5. (20 pts) Consider three boxes and 12 balls, and exactly three of the balls are red. The first box fits any three balls, the second box fits any four balls and the third box will fit any five balls. Let us randomly put the balls into the boxes.

Show that the probability that all three red balls end up in the same box is $6.\widehat{81}\%$.

Solution. We count exactly in how many ways this can happen and divide it by the total number of arrangements for the balls. Since we have 3 indistinguishable red balls and 9 indistinguishable balls of other colors the total number of arrangements is $\binom{12}{3}$. Now if the 3 red balls fall into the first box, they can do so in $\binom{3}{3}$ whereas the other 9 balls land on the other boxes in exactly one way since they are indistinguishable. Likewise if the 3 balls fall in the second box and third box they do so in $\binom{4}{3}$ and $\binom{5}{3}$ ways respectively. We compute the fraction

$$\frac{\binom{3}{3}+\binom{4}{3}+\binom{5}{3}}{\binom{12}{3}},$$

and it is immediately verified that the probability is indeed $\approx 6.82\%$. $\square$

Problem 6. (20 pts) By definition, a poker hand is a set of 5 cards from a standard French deck of 52 cards[1]. Solve the following two problems:

(a) A poker hand is said to be a *four of a kind* if it has four cards with the same value. For instance, four *sevens*, four *Queens* or four *Aces*. Compute the probability of drawing a four of a kind hand.

(b) A poker hand is said to be a *full house* if it has three cards with the one value, and two cards with a second value. For instance, three *sevens* and two *Queens*, or three *Aces* and two *fives*. Compute the probability of drawing a full house.

Conclude why in the game of poker, a four of a kind beats a full house.

[1]We shall consider a standard French playing deck, which includes thirteen ranks in each of the four French suits. Thus, each suit will have 13 possible values.

Solution. **Part (a)** The total number of ways to draw 5 cards is $\binom{52}{5}$. To draw 4 of a kind first we choose which of the 13 values we have. There is a unique configuration of suits for these four cards, since all four suits appear. Regarding the remaining card, we have 12 possible choices for the value and 4 for the suit, thus there are $12 \cdot 4$ choices for that last card. Thus the probability is

$$\frac{13 \cdot 12 \cdot 4}{\binom{52}{5}}.$$

Part (b) Let us choose first the value for the three cards, which gives 13 choices, and there are $\binom{4}{3}$ configurations of suits. For the pair, there are 12 possible values and $\binom{4}{2}$ suits. Hence, the probability is:

$$\frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}}.$$

Problem 7. (20 pts) The following Problem comes from the game of Treize, featuring prominently in one of the earliest probability books, by P.R. de Montmort[2]. The game can be described as follows.

Consider the 13 cards of a given suit, so there are 13 different values, and draw them face down to the table. Each player calls out a number between 1 and 13 and turns one card face up. A player loses if the called value coincides with the card value. If no player loses by the time the last card is turned face up, then the dealer loses.

(a) Compute the probability that the dealer loses.

(b) Suppose that we played with 1982 distinct cards with values from 1 to 1982. Estimate the probability that the dealer loses.

Solution. **Part (a)** Each player is given a card from 1 to 13 and says a (different) number from 1 to 13. We can think of this as a permutation of 13 elements. The dealer loses precisely when there are no fixed points. By Problem 7 in Problem Set 2, we know that there are :

$$13!\left(\sum_{k=0}^{13} \frac{(-1)^k}{k!}\right)$$

Dividing by the total number of permutations, namely 13!, we obtain the desired probability:

$$\sum_{k=0}^{13} \frac{(-1)^k}{k!}.$$

Part (b) By the same reasoning the desired probability is

[2]P. R. de Montmort, Essay d'Analyse sur des Jeux de Hazard, 2d ed. (Paris: Quillau, 1713), with the First Edition actually dating back to 1708. This problem is the First of the Exercises in Part 4 of this book, it reads "Déterminer generalement quel est à ce jeu l'avantage de celui qui tient les cartes.", i.e. "Find the expected value to the dealer when playing Treize.".

$$\sum_{k=0}^{1982} \frac{(-1)^k}{k!}.$$

Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, with 1982 cards the above sum can be approximated by e^{-1}.

Problem 8. Consider a squared $(n \times n)$-grid, and staircase walk that lies below, but is allowed to touch, the diagonal Δ formed by the points

$$\Delta = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y, 0 \leq x, y \leq n\}.$$

By definition, a *Dyck path* will be any such staircase walk below the diagonal. Find the probability that a staircase walk is a Dyck path.

Solution. To solve this problem we need only compute the number of Dyck paths and total staircase path as its ratio gives the probability. Since in total there are n up and n right moves to allocate it is easy to see that in total we have $\binom{2n}{n}$ staircase paths.

Now we claim that the number of Dyck paths is given by the Catalan numbers namely $C_n = \frac{1}{n+1}\binom{2n}{n}$. Rotating the picture by 45 degrees and reflecting it note that Dyck paths of length n simply random walks on $\mathbb{Z}$ that stay above the x-axis.

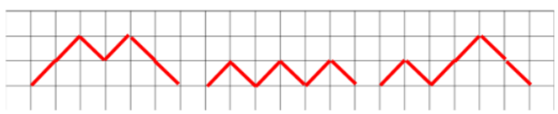

FIGURE 1. Dyck paths as walks above the x-axis.

We will show that the number of Dyck paths satisfies the same recursive relationship as the Catalan numbers:

$$C_{n+1} = \sum_{k=1}^{n+1} C_k C_{n-k}.$$

First of all, by a simple parity argument note that we only hit the x axis at even lattice points. In fact, for any given walk let $(2k, 0)$ be the point at which it hits the x axis for the first time. Then it is clear that there are $C_k C_{n-k}$ such walks; that is C_k to get to $(2k, 0)$ from the origin and C_{n-k} to get to $(2n, 0)$ from $(2k, 0)$. Since the first hitting point of the x axis can be any of $\{(2, 0), (4, 0)...(2n, 0)\}$ the total number of Dyck walks of length $n + 1$ is given by the sum

$$\sum_{k=1}^{n+1} C_k C_{n-k}.$$

So the number of Dyck paths follows the same recurrence as the Catalan the numbers. They are indeed the same by noting that $C_1 = 1 = \#\{\text{Dyck paths of length } 1\}$.

Therefore the probability that a random staircase path of length n is Dyck is:

$$\frac{C_n}{\binom{2n}{n}} = \frac{1}{n+1}.$$