

MAT 145: PROBLEM SET 4

DUE TO FRIDAY FEB 22

ABSTRACT. This problem set corresponds to the sixth week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Friday Feb 15 and is due Friday Feb 22 at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the material covered during the sixth week of lectures. In particular, we would like to become familiar with examples of graphs, Euler and Hamiltonian walks and circuits and the degree-sum formula.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use “Discrete Mathematics: Elementary and Beyond” by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Show that there exists no graph $G = (V, E)$ with $|V| = 48$ vertices such that the degrees of 30 of the vertices are 16, the degree of 15 vertices is 9 and the degree of the remaining 3 vertices is 12.

Problem 2. Let $G = (V, E)$ be a connected graph, an edge $e \in E$ is a *cut-edge* if $G \setminus \{e\}$ is disconnected. Show that if G admits an Euler circuit, then there exist **no** cut-edge $e \in E$.

Problem 3. (20 pts) For each of the three graphs in Figure 1, determine whether they have an Euler walk and/or an Euler circuit. Justify your answer, i.e. if an Euler walk or circuit exists, construct it explicitly, and if not give a proof of its non-existence.

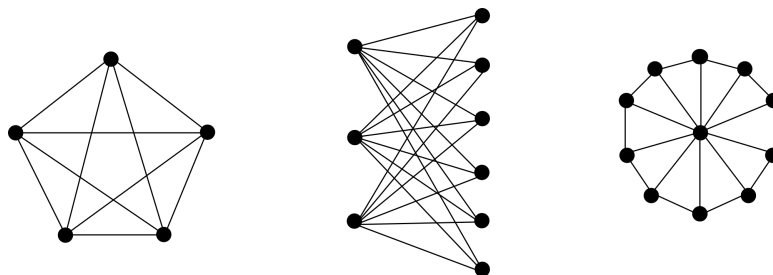


FIGURE 1. The three graphs for Problem 3 K_5 , $K_{3,6}$ and the Wheel W_{10} .

Problem 4. (20 pts) Let $n, m \in \mathbb{N}$ be two natural numbers. Let K_n be the complete graph in n vertices, and $K_{n,m}$ the complete bipartite graph in n and m vertices¹. See Figure 2 for two Examples of such graphs.

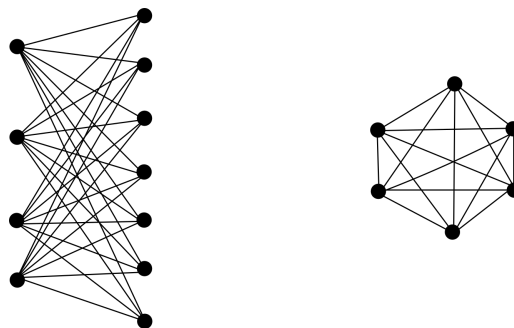


FIGURE 2. The $K_{4,7}$ on the Left and K_6 on the Right.

- (a) Determine the number of edges of K_n , and the degree of each of its vertices. Given a necessary and sufficient condition on the number $n \in \mathbb{N}$ for K_n to admit an Euler circuit.
- (b) Determine the number of edges of $K_{n,m}$, and the degree of each of its vertices. Given a necessary and sufficient condition on the numbers $n, m \in \mathbb{N}$ for $K_{n,m}$ to admit an Euler circuit.
- (c) Show that the complete bipartite graph $K_{n,m}$ admits a Hamiltonian cycle if and only if $n = m$.

Problem 5. (20 pts) Show that there are exactly three connected graphs with 4 vertices or less which admit an Euler circuit. In addition, list four different connected graphs with 5 vertices which admit Euler circuits, and find five different connected graphs with 6 vertices with an Euler circuits.

¹In class, we also called $K_{n,m}$ the *utility* graph.

Problem 6. (20 pts) Decide whether the following statements are true or false. In case the statement is true, provide a proof, and if it is false, provide a counter-example.

- (a) The Petersen Graph does admit a Hamiltonian cycle.
See Figure 3 (Left) for a depiction of the Petersen graph.
- (b) The Herschel Graph does *not* admit a Hamiltonian cycle.
See Figure 3 (Right) for a depiction of this graph.

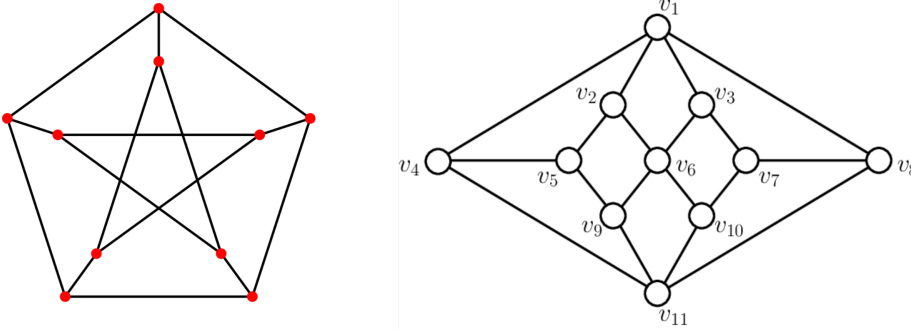


FIGURE 3. The Petersen Graph (Left) and the Herschel Graph (Right).

- (c) Every connected graph in 7 vertices admits a Hamiltonian cycle.
- (d) Let $G = (E, V)$ be a graph such that for all non-adjacent vertices $x, y \in V$

$$\deg(x) + \deg(y) \geq |V| - 1.$$

Then G is connected.

Problem 7. (20 pts) Let $n \in \mathbb{N}$ be a natural number and K_n the complete graph in n vertices. Show that K_n admits $(n - 1)!$ different² Hamiltonian cycles.

Problem 8. (De Bruijn Graphs) Consider the set $S(n)$ of binary sequences of length n , which is given by

$$S(n) := \{(s_1, \dots, s_n) : s_i \in \{0, 1\}, 1 \leq i \leq n\}.$$

Construct the directed³ graph B_n whose vertex set is $S(n)$, and such that each vertex $v = (s_1, \dots, s_n)$ has the following two edges going *out* of it, going to the vertices $(s_2, s_3, \dots, s_n, 0)$ and $(s_2, s_3, \dots, s_n, 1)$. Show that B_n admits a (directed) Euler circuit for all $n \in \mathbb{N}$.

²A Hamiltonian cycle is considered to be an ordered list of all vertices, where only *adjacent* vertices are allowed to be consecutive. Such a list is only considered up to cyclic ordering.

³A graph is directed if the edges are oriented, i.e. each edge goes from a vertex to another vertex, that is, all the edges are directed from one vertex to another.