MAT 145: PROBLEM SET 5

DUE TO FRIDAY MAR 1

Abstract. This problem set corresponds to the sixth week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Tuesday Feb 19 and is due Friday March 1st at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the material covered during the seventh week of lectures. In particular, we would like to become familiar with tree graphs, labeled and unlabeled trees, Prüfer codes and Cayley's Theorem.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. There are six different, i.e. mutually non-isomorphic, trees with six vertices. Draw these six trees.

Hint: For further practice, there is a unique tree with each of exactly 1, 2 or 3 vertices, there are two trees with 4 vertices and three trees with 5 vertices.

Problem 2. A *leaf* is a vertex of degree 1. Prove that if a tree $T = (V, E)$ has a vertex of degree d, then it has at least d leaves.

Problem 3. (20 pts) Let $G = (V, E)$ be a connected graph, a connected subset $T \subseteq E$ is said to be a *spanning tree* for G if it satisfies the following two properties:

(i) Every vertex of G belong to an edge of T,

(ii) The edges in T form a tree.

Solve the following two problems.

(a) Find a spanning tree for each of the graphs in Figure 1. Are they unique in these cases ?

(b) Show that every connected graph has at least one spanning tree.

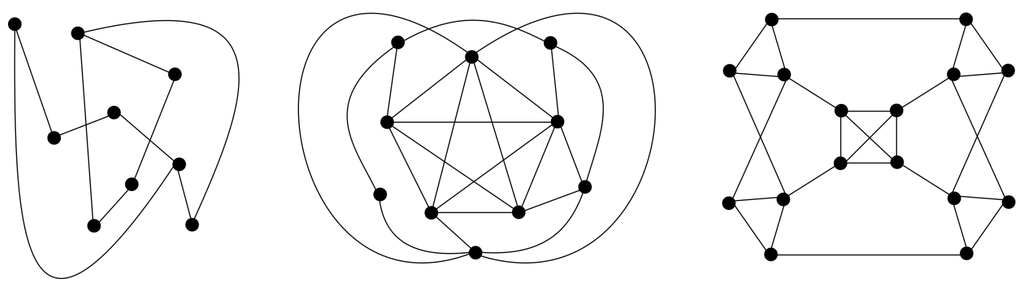

FIGURE 1. The three graphs for Part (a), find a spanning tree.

Problem 4. (20 pts) Let $n, m \in \mathbb{N}$ be two natural numbers.

(a) Show that the number of spanning trees of the complete graph K_n is n^{n-2}.

(b) (8.5.12) A (n, m)-dumbbell graph is constructed by considering the complete graph K_n on n vertices, the complete graph K_m on m nodes, and connecting them by a single edge. Figure 2 depicts the case $n = 4$ and $m = 5$. Find the number of spanning trees of a dumbbell graph.

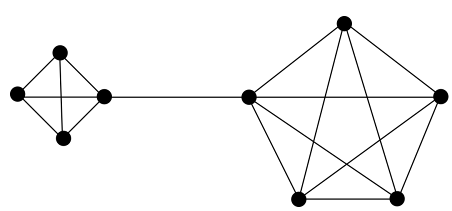

FIGURE 2. The $(4, 5)$-dumbbell graph.

Problem 5. (20 pts) The Prüfer correspondence[1] allows us to prove Cayley's Theorem. This problem gives direct practice on that correspondence.

(a) Draw the five labeled trees corresponding to the following five Prüfer codes:

$$\{4,4,4\}, \quad \{0,0,0\}, \quad \{1,2,4\}, \quad \{0,3,6,2,5\}, \quad \{2,3,4,5,6\}.$$

(b) Find the Prüfer code of the five labeled trees depicted in Figure 3.

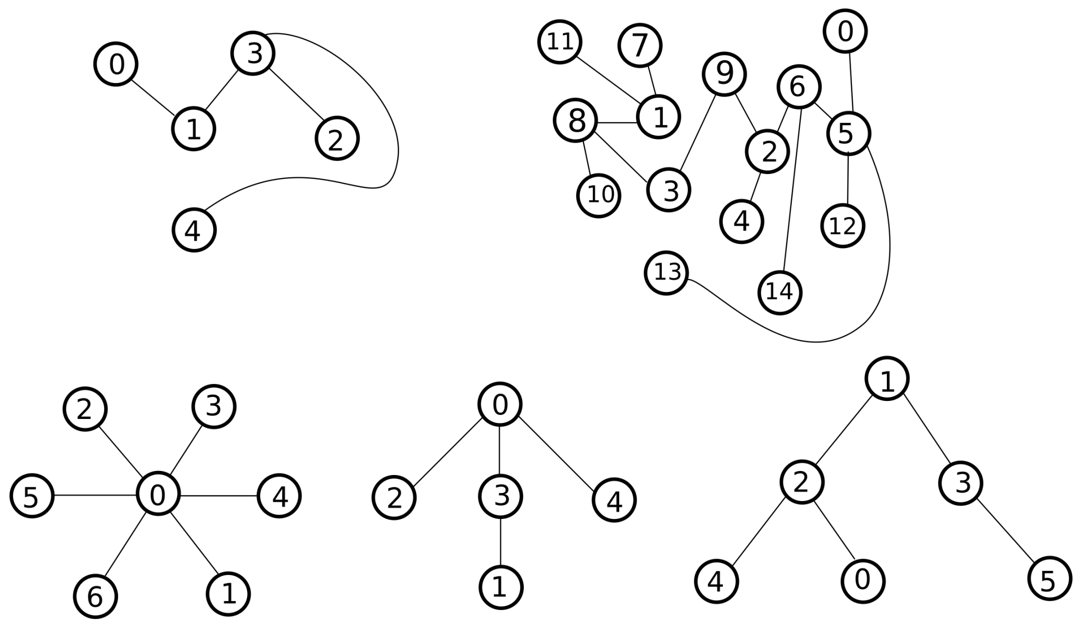

FIGURE 3. The five trees for Problem 5 Part (b).

[1]See Section 8.4 and the Lecture on Friday February 22nd.

Problem 6. (20 pts) A *binary* tree is a rooted tree in which each vertex has at most two children, which are ordered, and are referred to as the *left child* and the *right child*. For instance, there are 2 binary trees in two vertices and 5 binary trees in three vertices, depicted in Figure 4. Let $n \in \mathbb{N}$ be a natural number.

(a) Show that the number of binary trees in n vertices is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Hint: Notice that the numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ *satisfy the recursion*

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

(b) Establish a bijection between binary trees in n vertices and triangulations of regular convex $(n+2)$-gon. How many triangulations does an octagon have ?

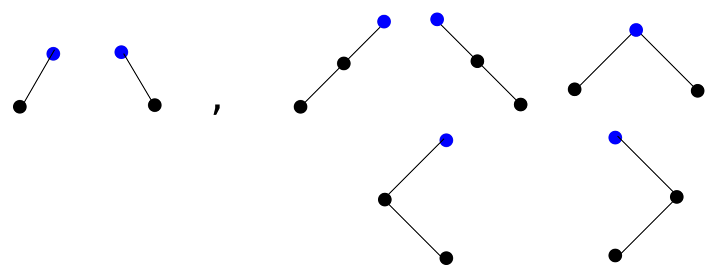

FIGURE 4. The two rooted binary trees with two vertices (Left) and the five rooted binary trees with three vertices (Right).

Problem 7. (20 pts) Solve the following two problems.

(a) Let $n_1, n_2, n_3 \ldots, n_d$ be a sequence of d natural numbers and $d \geq 2$. Show that there exists a tree $T = (V, E)$ with vertex degrees exactly $n_1, n_2, n_3 \ldots, n_d$ if and only if $n_1 + n_2 + n_3 + \ldots + n_d = 2d - 2$.

(b) Let $T = (V, E)$ be a tree with no vertices of degree 2. Show that there are more leaves[2] than non-leaves.

Problem 8. Let $G = (V, E)$ be a connected graph. Show that G is a tree if and only if any three pairwise vertex-intersecting paths in G have a common vertex.

[2]By definition, as explained in Problem 2 above, a *leaf* is a vertex of degree 1.