MAT 145: PROBLEM SET 6

DUE TO FRIDAY MAR 8

Abstract. This problem set corresponds to the eight week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Thursday Feb 28 and is due Friday March 8st at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the material covered during the eighth week of lectures. In particular, we would like to study through bipartite graphs, perfect matchings and planar graphs. This includes Hall's Theorem and Euler's Formula as two of the main results.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $n \in \mathbb{N}$ be a natural number. Show that the complete bipartite graph $K_{n,n}$ admits $n!$ perfect matchings.

Problem 2. Let G be connected graph with 12 vertices. Suppose that it admits an planar embedding $G \subseteq \mathbb{R}^2$ dividing the plane $\mathbb{R}^2$ into 20 faces. How many edges does G have ?

Problem 3. (20 pts) Solve the following three problems.

(a) (10 pts) Show that the three connected graphs in Figure 1 are *not* bipartite, and find a perfect matching in the first and third graphs.

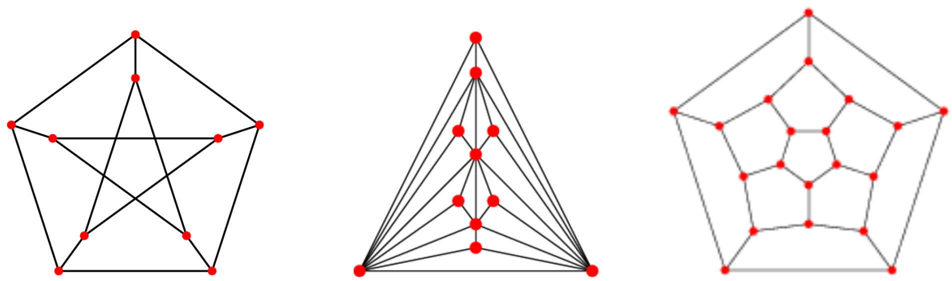

FIGURE 1. The three graphs for Problem 3.(a).

(b) (5 pts) Prove that the three connected graphs in Figure 2 do not admit any perfect matching. (Note the the second and third graphs are $K_{3,2}$ and $K_{2,5}$.)

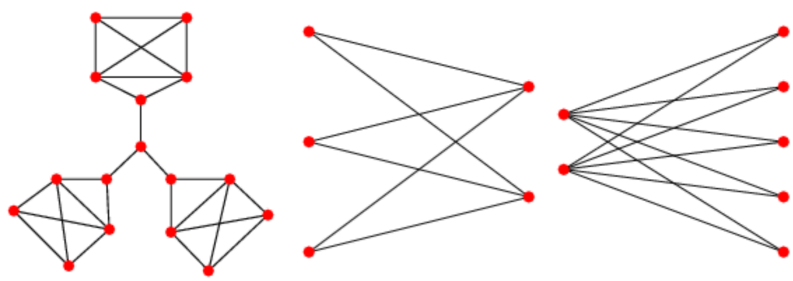

FIGURE 2. The three graphs for Problem 3.(b).

(c) (5 pts) Let $n, m \in \mathbb{N}$ be two natural numbers. Show that the complete bipartite graph $K_{n,m}$ admit a perfect matching if and only if $n = m$.

Problem 4. (20 pts) Solve the following three parts.

(a) (10 pts) Let $G = (V, E)$ be a connected bipartite graph. Suppose that every vertex $v \in V$ has the same degree. Show that G admits a perfect matching.

(b) (5 pts) Give an example of a connected graph G such that every vertex $v \in V$ has the same degree, but G does *not* admit a perfect matching.

(c) (5 pts) For any $n \in \mathbb{N}$, a natural number, with $n \geq 3$, give an example of a connected bipartite graph $G = (V, E)$ with $|V| = n$ and G does *not* admit a perfect matching.

Problem 5. (20 pts) Let $r, n \in \mathbb{N}$ be two natural numbers with $r \leq n$. An $r \times n$ matrix M consisting of r rows and n columns is said to be a **Latin rectangle** of size (r, n), if all the entries M_{ij} belong to the set $\{1, 2, 3, \ldots, n\}$, for $1 \leq i \leq r$, $1 \leq j \leq n$, and the same number does *not* appear twice in any row or in any column. By definition, a Latin square is a Latin rectangle of size (n, n), i.e. a Latin rectangle with $r = n$.

For instance, with $r = 3$, $n = 5$, the following two matrices are Latin rectangles

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 4 & 5 & 2 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$$

whereas the following two matrices are *not* Latin rectangles

$$\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 4 & 5 & 1 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

(a) (5 pts) Show that two different Latin squares of size 3×3 exist. In addition, construct a Latin square of size 4×4.

(b) (10 pts) Let M be a Latin rectangle of size (r, n) with $r < n$. Show that it is possible to add a row to M such that the resulting $(r + 1, n)$ rectangle is also a Latin rectangle.

Hint: Build a bipartite graph $G(M) = (A \cup B, E)$ from the Latin rectangle M according to the possible numbers (vertices in A) which can go into each column entry (vertices in B) of the new row. Then use Hall's Theorem to prove that $G(M)$ admits a perfect matching.

(c) (5 pts) Show that any Latin rectangle of size (r, n) can be completed, by adding rows, to a Latin square of size (n, n).

Problem 6. (20 pts) Consider a standard French deck of cards, with 4 suits and 13 values per suit, and shuffle it randomly. Deal 13 different piles, each pile containing 4 cards, the cards being face up. Show that you can **always** select exactly one card from each pile such that the 13 selected cards have the values $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$.

Hint: Translate the problem into a problem about perfect matchings on graphs, and then apply Hall's Theorem.

Problem 7. (20 pts) Solve the following two parts.

(a) For each of the six connected graphs in Figure 3, decide whether they are planar or not. If a graph is planar, draw a planar embedding. Else, give an argument showing its non-planarity.

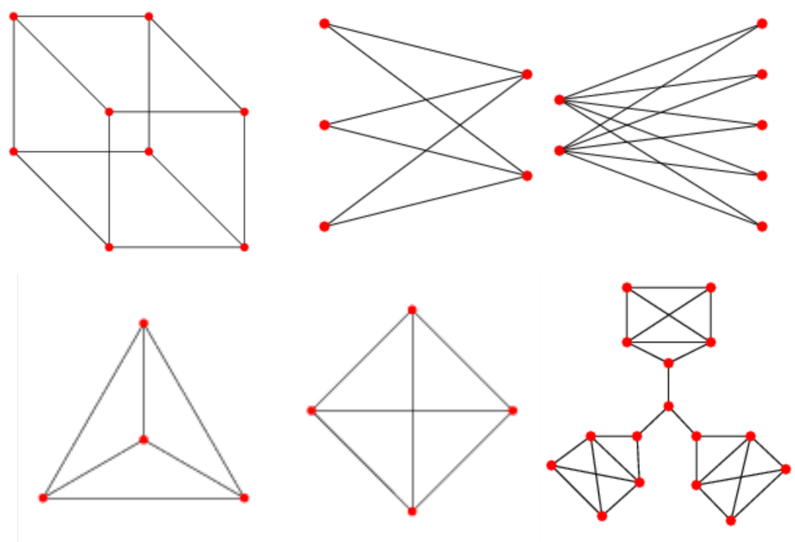

FIGURE 3. The six graphs for Problem 7.(a).

(b) Let $n \in \mathbb{N}$, prove that K_n is planar if and only if $n \leq 4$.

Problem 8. Let $n, m \in \mathbb{N}$ be two natural numbers, $n \leq m$.

(a) Show that the complete bipartite graph $K_{n,m}$ is planar if and only if $n \leq 2$.

(b) Characterize in terms of $n, m \in \mathbb{N}$ which (n, m)-dumbbell graphs are planar.