MAT 145: PROBLEM SET 6

DUE TO FRIDAY MAR 8

ABSTRACT. This problem set corresponds to the eighth week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Friday Mar 1 and is due Friday Mar 8 at the beginning of the class at 9:00am.

Information: These are the solutions for the Problem Set 5 corresponding to the Winter Quarter 2019 class of MAT 145 Combinatorics, with Prof. Casals and T.A. A. Aguirre. The Problems are written in black and the solutions in blue.

Problem 1. Let $n \in \mathbb{N}$ be a natural number. Show that the complete bipartite graph $K_{n,n}$ admits $n!$ perfect matchings.

Solution Let $K_{n,n} = A \cup B$. Take the first node in A, we can connect it to n different nodes in B. For the second node in A we must exclude the node in B that has been connected to the first node in A so we have $n-1$ options. Continuing this procedure we obtain $n!$ distinct perfect matchings.

Problem 2. Let G be connected graph with 12 vertices. Suppose that it admits an planar embedding $G \subseteq \mathbb{R}^2$ dividing the plane $\mathbb{R}^2$ into 20 faces. How many edges does G have ?

Solution Euler's formula $v - e + f = 2$ gives us that there are 30 edges.

Problem 3. (20 pts) Solve the following three problems.

(a) (10 pts) Show that the three connected graphs in Figure 1 are *not* bipartite, and find a perfect matching in the first and third graphs.

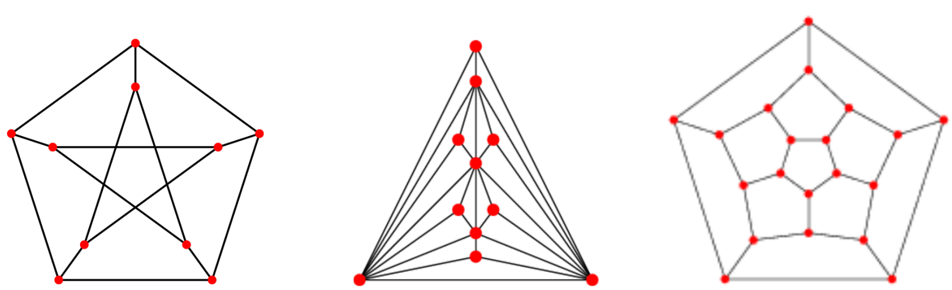

FIGURE 1. The three graphs for Problem 3.(a).

(b) (5 pts) Prove that the three connected graphs in Figure 2 do not admit any perfect matching. (Note the the second and third graphs are $K_{3,2}$ and $K_{2,5}$.)

(c) (5 pts) Let $n, m \in \mathbb{N}$ be two natural numbers. Show that the complete bipartite graph $K_{n,m}$ admit a perfect matching if and only if $n = m$.

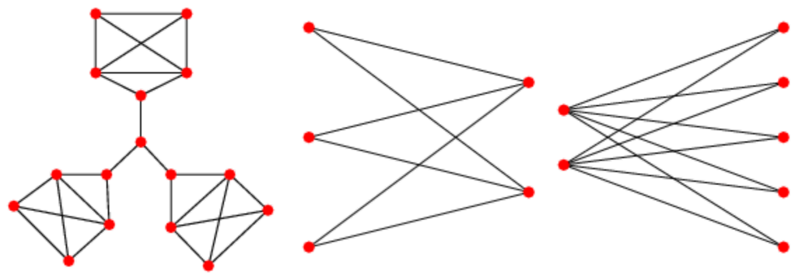

FIGURE 2. The three graphs for Problem 3.(b).

Solution

(a) Recall Theorem in the book that says that any graph that contains an odd length cycle has no 2-colorings (i.e. perfect matchings). The first and thirds graphs contain a 5-cycle while the second one contains a 3-cycle.

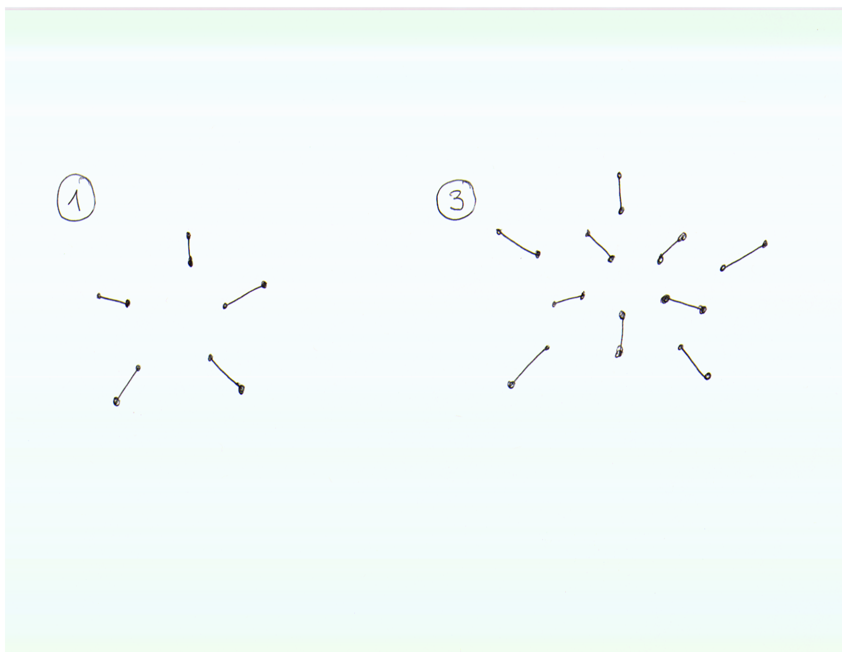

FIGURE 3. Perfect matchings for the first and third graphs.

(b) Any perfect matching of these graph must contain the center node and therefore exactly one of the edges coming out of it. This implies that such perfect matching must contain a perfect matching of the following subgraph:

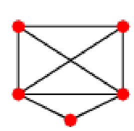

FIGURE 4. Onethe subgraphs attached to the center node.

Now this subgraph contains no perfect matching because it has an odd number of vertices. The other two graphs are $K_{3,2}$ and $K_{2,5}$; they do not contain a perfect matching by section (c) of this exercise.

(c) If $n = m$ we have in fact $n!$ perfect matchings by Problem 1. Conversely suppose that there are no perfect matchings in $K_{n,m}$ then by Hall's theorem either $m \neq n$ or for every subset S of k nodes of A is connected than less than k nodes in B. If $n \neq m$ we are done. Otherwise suppose that the k nodes of S reach $|N(S)| < k$ nodes in B where $N(S)$ is the neighboring set of S. The number of outgoing edges from S ($m \times |S|$) and incoming edges into $N(S)$ ($n \times |N(S)|$) have to match so $|S| \times m = |N(S)| \times n$; equivalently $\frac{n}{m} = \frac{|A|}{|N(A)|} > 1$ which shows $n \neq m$

Problem 4. (20 pts) Solve the following three parts.

(a) (10 pts) Let $G = (V, E)$ be a connected bipartite graph. Suppose that every vertex $v \in V$ has the same degree. Show that G admits a perfect matching.

(b) (5 pts) Give an example of a connected graph G such that every vertex $v \in V$ has the same degree, but G does *not* admit a perfect matching.

(c) (5 pts) For any $n \in \mathbb{N}$, a natural number, give an example of a connected bipartite graph $G = (V, E)$ with $|V| = n$ and G does *not* admit a perfect matching.

Solution

(a) If $V = A \cup B$ we first show that $|A| = |B|$. Suppose the degree of every vertex is k. Then there are $k|A|$ outgoing edges from A. Since each vertex in B has exactly k incoming edges we have that $|A| = B$. Now let's consider a subset $S \subset A$ of m nodes. Denote by $N(S)$ the subset of B reached by outgoing edges from S. The number of outgoing edges from S has to match the incoming edges to $N(S)$. To show the existence of a perfect matching by Hall's theorem it suffices to show that $|N(S)| \geq m$. Suppose by contradiction that $|N(S)| < m$. We have $m \times k$ outgoing edges from A but $|N(S)| \times k < m \times k$ incoming edges into $N(S)$.

(b) By part (a) we must look for examples that are *not* bipartite. Consider K_5, it is clearly connected and all its vertices have degree 4 however it has no perfect matching because it has an odd number of nodes.

(c) By part (a) we must for examples where *not all* vertices have the same degree. For a given $n \in \mathbb{N}$ consider l, m such that $l \neq m$ and $l + m = n$. Then $K_{l,m}$ is connected but by exercise 3c it contains no perfect matching.

Problem 5. (20 pts) Let $r, n \in \mathbb{N}$ be two natural numbers with $r \leq n$. An $r \times n$ matrix M consisting of r rows and n columns is said to be a **Latin rectangle** of size (r, n), if all the entries M_{ij} belong to the set $\{1, 2, 3, \ldots, n\}$, for $1 \leq i \leq r$, $1 \leq j \leq r$, and the same number does *not* appear twice in any row or in any column. By definition, a Latin square is a Latin rectangle of size (n, n), i.e. a Latin rectangle with $r = n$.

For instance, with $r = 3$, $n = 5$, the following two matrices are Latin rectangles

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 4 & 5 & 2 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$$

whereas the following two matrices are *not* Latin rectangles

$$\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 4 & 5 & 1 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

(a) (5 pts) Show that two different Latin squares of size 3×3 exist. In addition, construct a Latin square of size 4×4.

(b) (10 pts) Let M be a Latin rectangle of size (r, n) with $r < n$. Show that it is possible to add a row to M such that the resulting $(r+1, n)$ rectangle is also a Latin rectangle.

Hint: Build a bipartite graph $G(M) = (A \cup B, E)$ from the Latin rectangle M according to the possible numbers (vertices in A) which can go into each column entry (vertices in B) of the new row. Then use Hall's Theorem to prove that $G(M)$ admits a perfect matching.

(c) (5 pts) Show that any Latin rectangle of size (r, n) can be completed, by adding rows, to a Latin square of size (n, n).

Solution

(a) Consider the following two $(3,3)$ Latin squares:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

For a Latin square of size $(4,4)$ consider:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

(b) Let A be the set of columns of the Latin rectangle (note that $|A|$=n) and $B = \{1, 2, 3, ..n\}$ (so that we also have $|B| = n$). For the edge set E we will connect the vertices in A and B in the following way. Take a column c_i, there are exactly $n - r$ integers $\{1, 2, 3, ..n\}$ that do not appear in c_i then we connect the node in A representing c_i to the nodes in B that represent the integers that do not appear in c_i. At this point note that if we show that there exists a perfect matching we can construct a new row such that the rectangle $(r+1, n)$ is still Latin. This is because by construction each column c_i was extended with an integer not appearing in it and the $r + 1$-th row now contains n distinct integers (because it is a perfect matching). The existence of the perfect matching will follow from Hall's theorem. The graph we constructed is clearly bipartite and $|A| = |B|$. Note as well that every vertex has degree $n - r$. This is clear for the vertices in A. For the vertices in B we note that each integer $\{1, 2, 3, ..., n\}$ appears r times in the Latin square (r, n) because it appears once in each row. Therefore there are exactly $n - r$ *columns* in which it does not appear and hence the degree of the nodes in B is $n - r$. Therefore if $S \subset A$ then there are $|S| \times (n - r)$ outgoing edges from S which have to match the number of incoming edges into $N(S)$. Since the degree of the vertices in $N(S)$ is also $(n-r)$ we have that $|S| = |N(S)|$ so by Hall's theorem there is a perfect matching.

(c) This is an immediate consequence of applying the result from 5b inductively. We iterate the construction of the new row $n - r$ times to get a Latin square of size (n, n)

Problem 6. (20 pts) Consider a standard French deck of cards, with 4 suits and 13 values per suit, and shuffle it randomly. Deal 13 different piles, each pile containing 4 cards, the cards being face up. Show that you can **always** select exactly one card from each pile such that the 13 selected cards have the values $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$.

Hint: Translate the problem into a problem about perfect matchings on graphs, and then apply Hall's Theorem.

Solution Consider the bipartite graph $(A \cup B, E)$ where A is the set of the 13 piles and $B = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$. Now as for the edges we connect each pile in A with the (up to 4) values in B that it contains. Clearly we have $|A| = |B|$. Moreover if we pick k nodes in A the k piles they represent will contain $4k$ cards so there must be at least k distinct values in them. Hence those k nodes are connected to at least k nodes in B and by Hall's theorem there is a perfect matching. The existence of the perfect mathcing precisely implies that we can select a card in each pile so that the 13 selected cards have the values $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$.

Problem 7. (20 pts) Solve the following two parts.

(a) For each of the six connected graphs in Figure 5, decide whether they are planar or not. If a graph is planar, draw a planar embedding. Else, give an argument showing its non-planarity.

(b) Let $n \in \mathbb{N}$, prove that K_n is planar if and only if $n \leq 4$.

Solution

(a) All the graphs are planar. See the figure below for their planar embeddings.

(b) First we show that if $n > 5$ K_n is not planar. For the case $n = 5$ Theorem 12.2.1 in the book shows K_5 is not planar. For $n \geq 6$ we note that K_n jas $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Theorem 12.2.2 in the book says that a planar graph on n nodes has at most $3n - 6$ edges. Finally note that $\frac{n(n-1)}{2} > 3n - 6$ if $n \geq 6$ so there are no complete planar graphs on 5 or more edges. Conversely, for $n = 1, 2, 3$ we have that K_n are a single node, a straight line connecting two

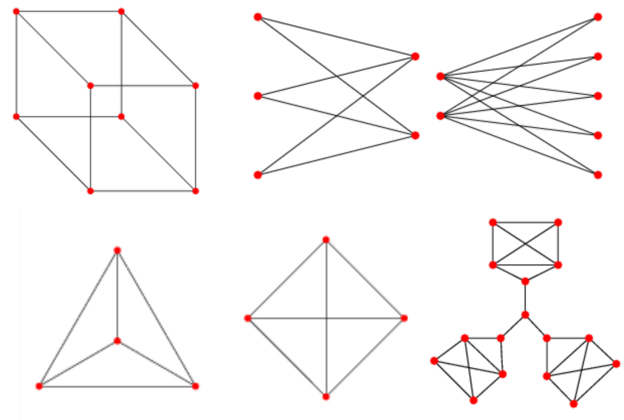

FIGURE 5. The six graphs for Problem 7.(a).

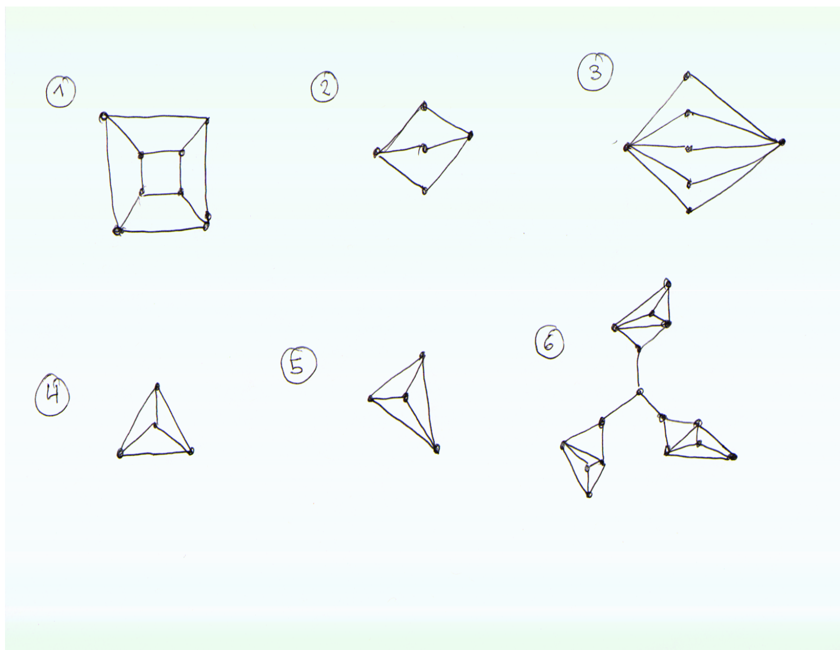

FIGURE 6. Planar embeddings for the six graphs in Problem 7.(a).

nodes and a triangle respectively; all of which which are clearly planar. We can see that K_4 is planar, in fact it is the second graph in the bottom row of section (a) of this problem.

Problem 8. Let $n, m \in \mathbb{N}$ be two natural numbers, $n \leq m$.

(a) Show that the complete bipartite graph $K_{n,m}$ is planar if and only if $n \leq 2$.

(b) Characterize in terms of $n, m \in \mathbb{N}$ which (n, m)-dumbbell graphs are planar.

Solution

(a) Take a look at the second and third graphs in Problem 7. (a). ($K_{3,2}$ and $K_{2,5}$). Their planar embeddings involved taking the partition of size 2 and moving each of the two nodes to the right and left of the other partition respectively.

This is clearly possible for any m so if $n \leq 2$ then $K_{n,m}$ is planar. If $n \geq 3$ then at least two nodes in the partition of size n would have to be in the same side with respect to the other partition. Since we are dealing with the complete bipartite graph the edges coming from these two nodes will have to intersect before reaching the nodes in the other partition so $K_{n,m}$ is not planar for $n \geq 3$.

(b) If the complete graphs of size n, m are planar then certainly the dumbell graph is planar. The graphs K_n and K_m have planar embeddings on their own and the addition of the bridge edge will keep things planar. By the previous problem for $n, m \leq 4$ the (n, m)-dumbell graphs are planar. For the converse we use the following lemma: If G is planar then every subgraph of G is planar. Therefore, if either K_n or K_m fails to be planar the whole dumbell graph will not be planar. Again by the previous problem this is the case if either $n \geq 5$ or $m \geq 5$.