MAT 145: PROBLEM SET 7

DUE TO FRIDAY MAR 15

ABSTRACT. This problem set corresponds to the ninth week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Thursday March 7th and is due Friday March 15th at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the material covered during the ninth week of lectures. In particular, we would like to practice colorings of graphs, chromatic numbers, the chromatic polynomial and relationships to planarity.

Task: Solve Problems 1 through 8 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problem 8, it will not be graded but you can also learn from it. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelikán and K. Vesztergombi. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Show that a graph G admits a 2-coloring if and only if G has no odd cycles. Give an example of a graph H which does *not* admit a 2-coloring but admits a 3-coloring, i.e. a graph H with chromatic number $\chi(H) = 3$.

Problem 2. (Exercise 13.3.4) Let G be a graph, and suppose that every subgraph of G has a node of degree at most d. Show that G is $(d + 1)$-colorable.

Problem 3. (20 pts) Solve the following two parts on chromatic numbers.

(a) Show that the four graphs in Figure 1 have chromatic numbers $5, 3, 4, 7$, when reading left to right. Thus, show that they respectively have 5, 3, 4 and 7-colorings *and* this is the minimal number of colors that must be used.
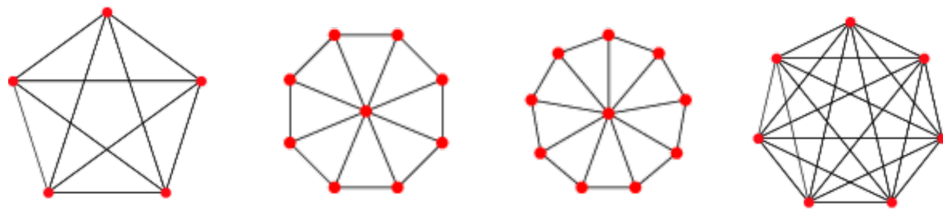

FIGURE 1. The four graphs for Problem 3.(a).

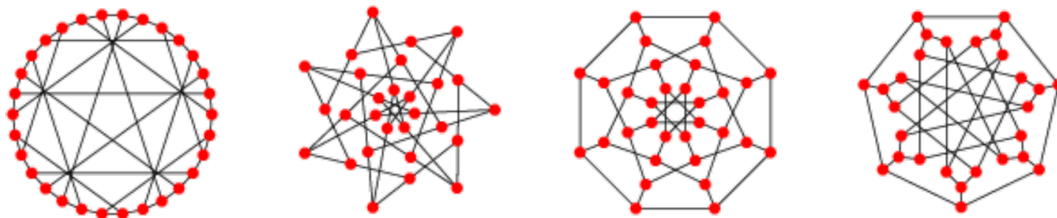(b) Compute the chromatic numbers of the four graphs in Figure 2.

FIGURE 2. Find the chromatic numbers for these graphs.

Problem 4. (20 pts) Let $G = (V, E)$ be a finite graph, and define the *clique number* $\omega(G)$ as the largest natural number $n \in \mathbb{N}$ such that the complete graph K_n is contained inside of G.

(a) Show that $\omega(G) \leq \chi(G)$, that is, if a graph G contains a complete graph K_n, then it does *not* admits a k-coloring for $k \leq n - 1$.

(b) Prove that the inequality $\omega(G) \leq \chi(G)$ is not always an equality, i.e. find a graph G such that $\omega(G) < \chi(G)$.

Part.(b) thus shows that there are reasons for having high chromatic numbers, other than containing a complete graph. For instance, a graph might not be 5-colorable and yet not contain a K_5 either.

Problem 5. (20 pts) Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ a natural number, and suppose that G has chromatic number k. Show that the number of edges $|E|$ satisfies the inequality $|E| \geq \binom{k}{2}$.

Hint: Assume, for a proof by contradiction, that $|E| < \binom{k}{2}$.

Problem 6. (20 pts) Let G be a graph, and suppose that every pair of odd cycles in G have a common vertex. Solve the following two parts.

(a) (10 pts) Show that the chromatic number $\chi(G)$ satisfies $\chi(G) \leq 5$, i.e. G is 5-colorable[1].

(b) (10 pts) In case the inequality is sharp, that is, $\chi(G) = 5$, prove that G contains a complete graph K_5 in 5 vertices.

Problem 7. (20 pts) Solve the following two parts.

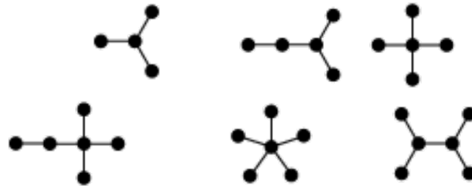(a) For each of the six trees in Figure 3, compute their chromatic polynomial.

FIGURE 3. The six graphs for Problem 7.(a).

(b) Show that the chromatic polynomial $\pi_T(x)$ of a tree $T = (V, E)$ is

$$\pi_T(x) = x(x - 1)^{|V|-1}.$$

(c) Prove that the chromatic polynomial $\pi_{K_n}(x)$ of the complete graph K_n in n vertices is given by the expression

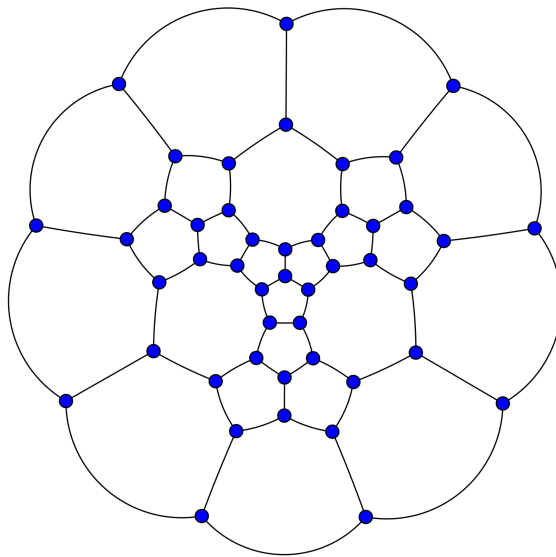$$\pi_{K_n}(x) = x(x - 1)(x - 2) \cdot \ldots \cdot (x - (n - 1)).$$

FIGURE 4. The graphs for Problem 8.(c).

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[1]Though possibly also k-colorable for $k \leq 5$.

Problem 8. Decide whether the following statements are *true* or *false*.

(a) The chromatic polynomial of the cyclic graph C_9 in 9 vertices is
$$\pi_{C_9}(x) = (x-1)^9 - (x-1).$$

(b) There exists a connected graph G such that its chromatic polynomial is
$$\pi_G(x) = x^4 - 4x^3 + 3x^2.$$

(c) The graph in Figure 4 admits a 4-coloring.
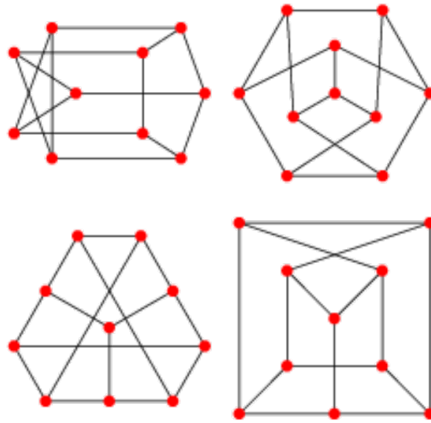
(d) The four graphs in Figure 5 all admit a 3-coloring.

FIGURE 5. The graphs for Problem 8.(e).

(e) Suppose that G has chromatic number $\chi(G) \geq 6$, then G is non-planar.

(f) Let G be a non-planar graph, then its chromatic number $\chi(G) \geq 3$.
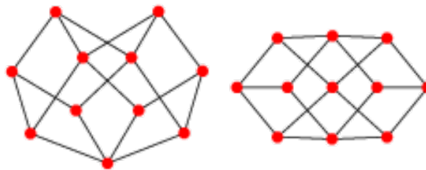
(g) The two graphs in Figure 6 do *not* admit 4-colorings.

FIGURE 6. The two graphs for Problem 8.(h).

(h) Let $G = (V, E)$ be a graph which is *not* a tree. Then the chromatic polynomial $\pi_G(x)$ of G is strictly less than $x(x-1)^{|V|-1}$.