MAT 145: PROBLEM SET 6

DUE TO FRIDAY MAR 8

Abstract. This problem set corresponds to the eighth week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Friday Mar 1 and is due Friday Mar 8 at the beginning of the class at 9:00am.

Information: These are the solutions for the Problem Set 5 corresponding to the Winter Quarter 2019 class of MAT 145 Combinatorics, with Prof. Casals and T.A. A. Aguirre. The Problems are written in black and the solutions in blue.

Problem 1. Show that a graph G admits a 2-coloring if and only if G has no odd cycles. Give an example of a graph H which does *not* admit a 2-coloring but admits a 3-coloring, i.e. a graph H with chromatic number $\chi(H) = 3$.

Solution. See the proof of Theorem 13.2.1 in page 201 of the textbook. For an example of a graph H with $\chi(H) = 3$ consider K_3. Clearly it can be colored with 3 colors since it only has 3 nodes. However, if we attempt a 2-coloring there will be a pair of adjacent vertices with the same color. □

Problem 2. (Exercise 13.3.4) Let G be a graph, and suppose that every subgraph of G has a node of degree at most d. Show that G is $(d + 1)$-colorable.

Solution. If this is the case then every node in G has degree at most d and Theorem 13.3.1 thus guarantees that G is $d + 1$-colorable. □

Problem 3. (20 pts) Solve the following two parts on chromatic numbers.

(a) Show that the four graphs in Figure 1 have chromatic numbers $5, 3, 4, 7$, when reading left to right. Thus, show that they respectively have 5, 3, 4 and 7-colorings *and* this is the minimal number of colors that must be used.

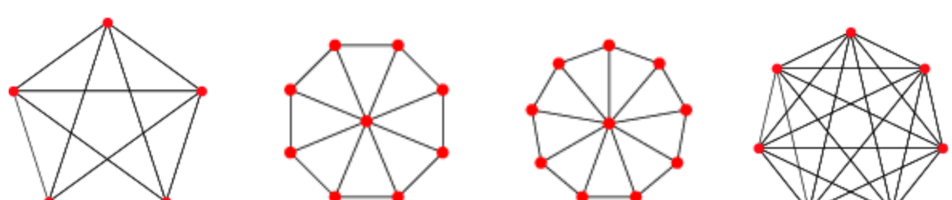

Figure 1. The four graphs for Problem 3.(a).

(b) Compute the chromatic numbers of the four graphs in Figure 2.

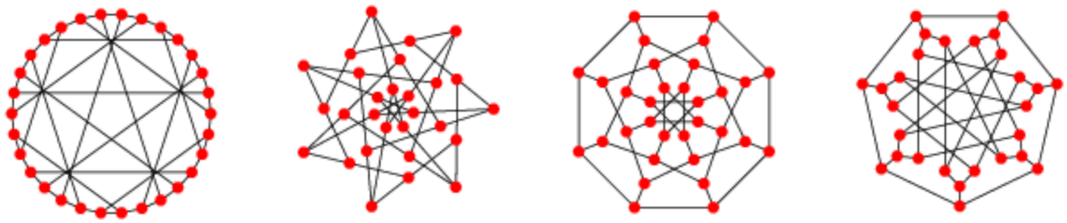

FIGURE 2. Find the chromatic numbers for these graphs.

Solution.

(a) We first prove as a lemma that the chromatic number of K_n is n. Since every node has degree $n-1$, Theorem 13.3.1 in page 203 guarantees the existence of an n-coloring. In fact since every pair of vertices is adjacent if we use strictly less than n colors some pair of vertices of the same color will be adjacent so we have $\chi(K_n) = n$. Observing that the first and last graphs are K_5 and K_7 respectively we have that their chromatic numbers are 5 and 7 (resp.).

For the middle graphs, we first show the lemma that all cyclic graphs C_n admit at least a 3-coloring. Cycles of even length are bipartite and in fact admit a 2-coloring. Earlier we saw that the chromatic number of K_3 is 3. Cycles of higher odd degree n can be constructed by inserting a "straight line" graph of degree $n-3$ in between any 2 nodes of K_3. Such stright line graph is 2-colorable and by ensuring that its end-nodes are attached to nodes in K_3 with different color the resulting cycle graph C_n is 3 colorable.

The second and third graphs are clearly planar so their chromatic number is at most 4. The second one is comprised by an outer even cycle (2-colorable) whith all its nodes adjacent to a center node which must have a different color so a 3 coloring exists and it is in fact minimal. For the third graph we have the same situation except that the outer cycle is of odd degree and therefore 3-colorable . Since the middle node has to be of a different color then the chromatic number of this graph is 4. □

(b) The first and the third graphs are bipartite, and thus their chromatic number is 2. Let us see this for the first one. Note that it is comprised by an even cycle, which admits a 2-coloring. Every other internal edge connects vertices that are an odd distance apart in the cycle and thus the 2-coloring of the outer cycle is indeed a 2-coloring of the whole graph. A similar argument shows that the third graph is bipartite. Regarding the second and fourth graphs, they contain odd cycles and thus they *cannot* be bipartite. Thus the chromatic number is bounded below by 3. By direct computation, i.e. finding a 3-coloring, we deduce that the chromatic number is 3 for the second and the fourth graphs. □

Problem 4. (20 pts) Let $G = (V, E)$ be a finite graph, and define the *clique number* $\omega(G)$ as the largest natural number $n \in \mathbb{N}$ such that the complete graph K_n is contained inside of G.

(a) Show that $\omega(G) \leq \chi(G)$, that is, if a graph G contains a complete graph K_n, then it does *not* admits a k-coloring for $k \leq n - 1$.

(b) Prove that the inequality $\omega(G) \leq \chi(G)$ is not an inequality, i.e. find a graph G such that $\omega(G) < \chi(G)$.

Part.(b) thus shows that there are reasons for having high chromatic numbers, other than containing a complete graph. For instance, a graph might not be 5*-colorable and yet not contain a* K_5 *either.*

Solution.

(a) Note that if H is a subgraph of G then we clearly have that $\chi(H) \leq \chi(G)$. Now in problem 3 (a) we showed that $\chi(K_n) = n$ which proves the desired result.

(b) Consider a cycle of odd length greater than 3, say 5. Then by the previous problem its chromatic number is 3 but not that it does *not* contain K_3 as a subgraph.

Problem 5. (20 pts) Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ a natural number, and suppose that G has chromatic number k. Show that the number of edges $|E|$ satisfies the inequality $|E| \geq \binom{k}{2}$.

Hint: Assume, for a proof by contradiction, that $|E| < \binom{k}{2}$.

Solution Assume that $|E| < \binom{k}{2}$ then there exist colors i, j such that there are no edges connecting nodes of colors i and j. This is because for every pair of colors to be connected we need at least $\binom{k}{2}$ edges (one for each pair of colors). However if nodes of colors i and j are not connected we can recolor all of them with color i, thus constructing a proper $k - 1$ coloring. This contradicts the assumption that the chromatic number of G is k.

Problem 6. (20 pts) Let G be a graph, and suppose that every pair of odd cycles in G have a common vertex. Solve the following two parts.

(a) (10 pts) Show that the chromatic number $\chi(G)$ satisfies $\chi(G) \leq 5$, i.e. G is, at least, 5-colorable.

(b) (10 pts) In case the inequality is sharp, that is, $\chi(G) = 5$, prove that G contains a complete graph K_5 in 5 vertices.

Solution

(a) Let C be the smallest odd cycle subgraph of G and let's color it using 3 colors (see problem 3). By the assumption that all odd cycles have a common vertex the subgraph $G \setminus C$ has no odd cycles; hence, by problem 1 it has a 2 coloring. These constitute colorings of C and $G \setminus C$ take separately. If their respective colors are disjoint, together they constitute a proper 5-coloring of G so $\chi(G) \leq 5$ as desired. $\square$

(b) Equivalently we can show that if $\chi(G) = 5$ and no K_5 is contained in G then our hypothesis that every pair of odd cycles intersects is contradicted. We can do this by taking at an odd cycle C (which exists since $\chi(G) \neq 2$; i.e it is not bipartite) and exhibiting another (disjoint) odd cycle in $G \setminus C$. Now suppose that not even K_4 is contained in G. We show that no vertex in $G \setminus C$ could possibly be connected to 3 vertices in C. If C is a triangle no vertex in $G \setminus C$ is adjacent to all vertices in C lest there be a subgraph isomorphic to K_4. Otherwise, let C be a minimal odd cycle and there are no triangles in the graph. Suppose suppose vertex K in $G \setminus C$ is connected to thee vertices in it, since C is odd at least one of the 3 distances (in the cycle) between two, say A and B, of those three nodes is odd. Then the triangle ABK is an odd cycle shorter than C (it takes 2 steps going back to A from B passing by K, completing the cycle would take more). This shows that every vertex in $G \setminus C$ is connected to at most 2 nodes in C so $\chi(G \setminus C) \geq \chi(G) - 2 = 3$. In other words a node of C might deprive of at most 2 choices in the coloring of a node of $G \setminus C$. But then $G \setminus C$ is not bipartite and there exists an odd cycle disjoint from C.

If we know that K_4 is a subgraph but K_5 is not likewise we will show there are two disjoint cycles. Since K_4 is contained the minimal odd cycle C is a triangle with vertices say a b and v. We will assume by contradiction that $G \setminus C$ is bipartite. There are at least two vertices d, e on the other side of the partition adjacent to all a, b and c. Now consider the triangle abd, we will construct another one disjoint from it. Since $G \setminus \{abd\}$ is not bipartite (or we are autoatically done because a disoing odd cycle exists) then as above we can find yet another vertex f adjacent to a, b and d. Then efc is another disjoint triangle. This argument thus shows that K_5 must be contained as a subgraph. $\square$

Problem 7. (20 pts) Solve the following two parts.

(a) For each of the six trees in Figure 3, compute their chromatic polynomial.

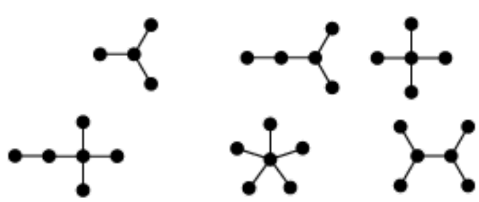

FIGURE 3. The six graphs for Problem 7.(a).

(b) Show that the chromatic polynomial $\pi_T(x)$ of a tree $T = (V, E)$ is

$$\pi_T(x) = x(x-1)^{|V|-1}.$$

(c) Prove that the chromatic polynomial $\pi_{K_n}(x)$ of the complete graph K_n in n vertices is given by the expression

$$\pi_{K_n}(x) = x(x-1)(x-2)\cdot \ldots \cdot (x-(n-1)).$$

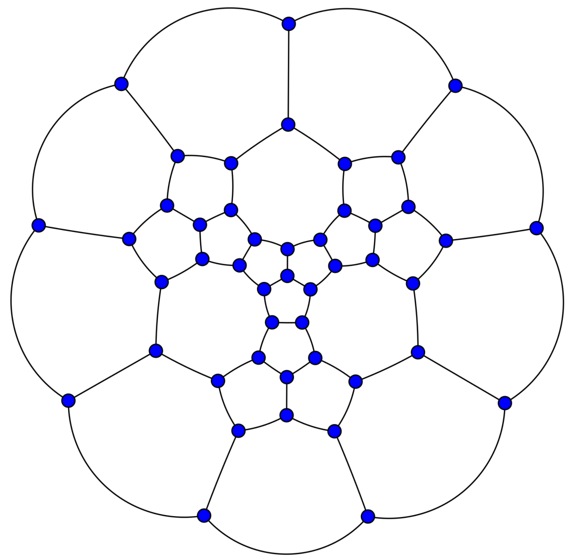

FIGURE 4. The graphs for Problem 8.(c).

Solution.

(a) By part (b) of this problem their chromatic polynomials are $x(x-1)^3$, $x(x-1)^4$ and $x(x-1)^4$ for the top row in left to right order and $x(x-1)^5$ for each tree in the bottom row. $\square$

(b) If $|V| = 2$ and G is a tree then it is K_2 and using x colors we can color it in $x(x-1)$ ways which proves the base case for induction. Now assume that for a tree with $|V| = n$ the chromatic polynomial is $x(x-1)^{n-1}$. Recall the procedure of growing a tree by adding one leaf at a time. If we have a tree on $n+1$ vertices, we remove a leaf to obtain a subgraph that is a tree on n vertices. This by assumption can be colored in $x(x-1)^{n-1}$ ways using x colors then the only restriction. The only restriction on the leaf is that it cannot have the same color as the vertex it is adjacent to. Thus the whole tree on n nodes can be colored in $x(x-1)^{n-1} \times (x-1) = x(x-1)^n$ ways. $\square$

(c) Since every pair of nodes are adjacent, in order to have a proper coloring using x color we can color the first node with x colors, the second one with $x-1$, the third one with $x-2$ etc. Therefore, the chromatic polynomial of K_n is $\pi_{K_n}(x) = x(x-1)(x-2) \cdot \ldots \cdot (x-(n-1))$. $\square$

Problem 8. Decide whether the following statements are *true* or *false*.

(a) The chromatic polynomial of the cyclic graph C_9 in 9 vertices is
$$\pi_{C_9}(x) = (x-1)^9 - (x-1).$$

(b) There exists a connected graph G such that its chromatic polynomial is
$$\pi_G(x) = x^4 - 4x^3 + 3x^2.$$

(c) The graph in Figure 4 admits a 4-coloring.

(d) The four graphs in Figure 5 all admit a 3-coloring.

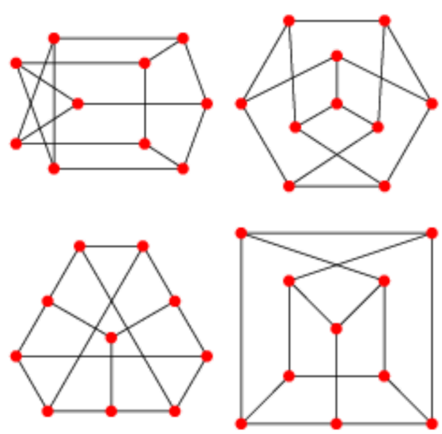

FIGURE 5. The graphs for Problem 8.(e).

(e) Suppose that G has chromatic number $\chi(G) \geq 6$, then G is non-planar.

(f) Let G be a non-planar graph, then its chromatic number $\chi(G) \geq 3$.

(g) The two graphs in Figure 6 do *not* admit 4-colorings.

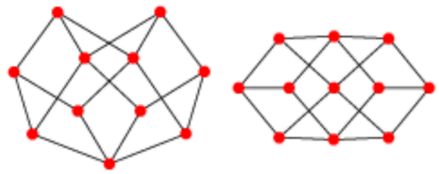

FIGURE 6. The two graphs for Problem 8.(g).

(h) Let $G = (V, E)$ be a graph which is *not* a tree. Then the chromatic polynomial $\pi_G(x)$ of G is strictly less than $x(x-1)^{|V|-1}$.

Solution

(a) True. By deletion contraction we have $\pi(C_9, x) = \pi(C_9 - e, k) - \pi(C_9 \cdot e, x) = \pi_{S_9}(x) - \pi_{C_8}(x)$. Note that deleting one edge we get the straight line graph S_9 (also known as P_9, which is a tree) and contracting one edge we get the smaller cycle C_8. Using the formula for the chromatic polynomial of trees and reiterating

$$\pi_{C_9}(x) = \pi_{S_9}(x) - \pi_{C_8}(x) = x(x-1)^8) - (\pi_{S_8}(x) - \pi_{C_7}(x)) = \cdots$$
$$= (x-1)^9 - (x-1)$$

A general formula exists for $\pi(C_n, x)$ and relies on this same inductive argument (see Practice Final 1).

(b) False. Note that $\pi_G(2) < 0$. This value is supposed to count the number of 2-colorings so it must be nonnegative.

(c) True. The graph is clearly planar and all planar graphs have a 4-coloring. (Theorem 13.4.1 pg 207).

(d) True. These four graphs are isomorphic. It's easiest to construct a three coloring on the bottom left one. Take the 9 vertices in the outer rim and color them with 3 colors in a periodic way. Then the middle vertex can be colored using any of the two colors that do not correspond to the color of its neighbors (the same for the three by periodicity). This gives a 3-coloring.

(e) True. Every planar graph admits a 4-coloring, so any graph with chromatic number strictly grater than 4 cannot be planar.

(f) False. Consider the bipartite graph $K_{3,4}$. Its chromatic number is 2 but it is non planar. See question 8 from Problem Set 6.

(g) False. The two graphs are isomorphic. For the graph on the right lets start with the top left rhombus and color it 3 colors. The rest of the graph consists of another 3 rhpmbuses which we can get by reflecting the upper left one across through horizontal and vertical lines with exactly the same colors. By construction everything so far admits a 3-coloring. Now the middle vertex connects to 4 vertices that have the same color (precisely by our construction by reflection), then it suffices to assign it a different color. Thus we have a 4-coloring (which we could've in fact improved to a 3 coloring).

(h) True. Any connected graph has a spanning tree. For every coloring of G we certainly have a coloring of its spanning tree T_G. Therefore

$$\pi_G(x) \leq \pi_{T_G}(x) = x(x-1)^{|V|-1}.$$