

University of California Davis
Combinatorics MAT 145

Name (Print): _____
Student ID (Print): _____

Midterm Examination
Time Limit: 50 Minutes

February 8 2019

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- (A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.
- (B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.
- (C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- (D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) Prove the following two statements.

(a) (10 points) Prove that for every $k, n \in \mathbb{N}$

$$k \binom{n}{k} = n \binom{n-1}{k-1}, \quad \text{if } k \leq n.$$

(b) (10 points) Show that for every $n \in \mathbb{N}$

$$\sum_{\substack{k=0, \\ k \text{ even}}}^n \binom{n}{k} = 2^{n-1}.$$

2. (20 points) Consider a 8×14 rectangular grid as depicted in Figure 1, with marked points A, B in the corners and P, Q in the interior part of the grid. A staircase walk is a path in the grid which moves only *right* or *up*.

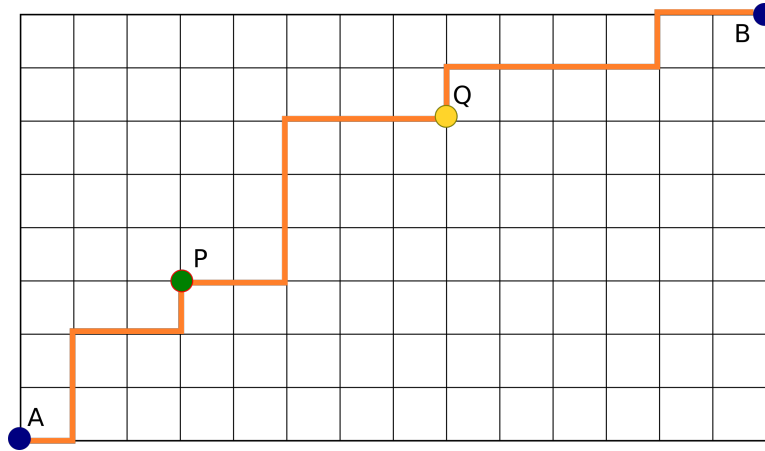


Figure 1: The 8×14 grid and a valid staircase walk from A to B passing through P and Q .

- (a) (10 points) Find the number of staircase walks from A to B **avoiding** the point P .
- (b) (10 points) How many staircase walks from A to B are there which either pass through the point P **or** pass through the point Q ?

3. (20 points) Let us assume that we have 35 presents and 6 people. These 35 presents will now be distributed to these 6 people, and we always assume that the order in which the presents are received does *not* matter.
- (a) (10 points) Suppose that the presents are *different*. Find the number of ways to distribute these 35 presents between these 6 people.
- (b) (10 points) Suppose that the presents are *equal*. Find the number of ways to distribute these 35 presents between these 6 people.

4. (20 points) Solve the following two questions.

(a) (10 points) Find the number of positive integers $n \in \mathbb{N}$, such that $1 \leq n \leq 1000$, which are **not** divisible by either 4, 10 or 15.

(b) (10 points) How many permutations of 5 elements do **not** fix any element ?

5. (20 points) Let us have a French deck of 52 cards, containing 4 suits with the 13 values

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}$$

in each of the four suits. We are dealt five cards from the deck, and these five cards constitute our hand. The order in which we hold the cards is immaterial. Solve the following two questions.

- (a) (10 points) A hand of five cards is said to be a *flush* if it contains five cards with the same suit. Find the probability of drawing a flush.

- (b) (10 points) A hand of five cards is said to have *two pairs* if it contains *exactly* two pairs of cards with the same value, the values of these pairs are *not* equal and the remaining card has a different value than the rest¹. For example, 2 *threes*, 2 *Queens* and 1 King is a hand with two pairs, whereas 4 *Queens* and 1 King is not.

Compute the probability of having a hand with two pairs.

¹In particular, there are no *three* cards with the same value in a hand with *two pairs*.