

University of California Davis  
Combinatorics MAT 145

Name (Print): \_\_\_\_\_  
Student ID (Print): \_\_\_\_\_

Midterm Examination  
Time Limit: 50 Minutes

February 8 2019

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- (A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.
- (B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.
- (C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- (D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) Prove the following two statements.
- (a) (10 points) Prove that for every  $k, n \in \mathbb{N}$

$$k \binom{n}{k} = n \binom{n-1}{k-1}, \quad \text{if } k \leq n.$$

**Solution.** Let us count the number of  $k$ -element subsets of a set of  $n$  elements with a preferred element in the subset. Equivalently, a team of  $k$ -people with a captain, or a committee of  $k$ -people with a chair.

The left hand side chooses this by selecting the team first, which gives  $\binom{n}{k}$ , and then then captain, which gives  $k$  choices. Since we are taking both decisions, these are  $k \cdot \binom{n}{k}$  outcomes. Alternatively, we could first choose the captain, giving  $n$  choices, and then the remaining  $(k-1)$ -members of the team, from the remaining  $(n-1)$  people. This latter method gives  $n \binom{n-1}{k-1}$ , which is the right hand side. Given that both sides are counting the cardinality of the *same* set, we conclude the desired equality.  $\square$

- (b) (10 points) Show that for every  $n \in \mathbb{N}$

$$\sum_{\substack{k=0, \\ k \text{ even}}}^n \binom{n}{k} = 2^{n-1}.$$

**Solution.** The left hand side counts the number of *even* subsets of a set  $X$  of  $n$ -elements. The right hand side also counts that. Indeed, consider an element  $e \in X$  and an *even* subset  $S$  of  $X$ . Then either  $S$  contains  $e$  or it does not. In the first case,  $S \setminus \{e\}$  is a subset of  $X \setminus \{e\}$ , and in the latter  $S$  is a subset of  $X \setminus \{e\}$ . Thus, *even* subsets of  $X$  are in bijections with all subsets of  $X \setminus \{e\}$ . Since the cardinality of  $X \setminus \{e\}$  is  $(n-1)$ , there must be exactly  $2^{n-1}$  subsets, which accounts for the right hand side.  $\square$

2. (20 points) Consider a  $8 \times 14$  rectangular grid as depicted in Figure 1, with marked points  $A, B$  in the corners and  $P, Q$  in the interior part of the grid. A staircase walk is a path in the grid which moves only *right* or *up*.

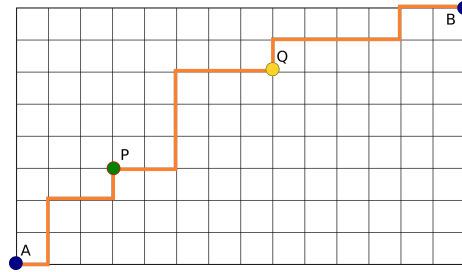


Figure 1: The  $8 \times 14$  grid and a valid staircase walk from  $A$  to  $B$  passing through  $P$  and  $Q$ .

- (a) (10 points) Find the number of staircase walks from  $A$  to  $B$  **avoiding** the point  $P$ .

**Solution.** A staircase walk from  $A$  to  $B$  will contain  $8 + 14$  units, and we have to choose going up 14 times. Thus there are  $\binom{22}{14}$  walks from  $A$  to  $B$ . Similarly, there are  $\binom{6}{3}$  walks from  $A$  to  $P$  and  $\binom{16}{5}$  walks from  $P$  to  $B$ . In consequence, the number of walks from  $A$  to  $B$  passing through  $P$  is  $\binom{6}{3} \cdot \binom{16}{5}$ .

The requested answer is obtained by subtracting all paths through  $P$  from the total number of paths, i.e.

$$\binom{22}{14} - \binom{6}{3} \cdot \binom{16}{5}.$$

- (b) (10 points) How many staircase walks from  $A$  to  $B$  are there which either pass through the point  $P$  **or** pass through the point  $Q$  ?

**Solution.** There are  $\binom{6}{3} \cdot \binom{16}{5}$  walks through  $P$ , as described above. Analogously, there are  $\binom{14}{6} \cdot \binom{8}{2}$  through  $Q$ . By the Inclusion-Exclusion Principle, the number of staircase walks from  $A$  to  $B$  are there which either pass through the point  $P$  **or** pass through the point  $Q$  will be

$$\binom{6}{3} \cdot \binom{16}{5} + \binom{14}{6} \cdot \binom{8}{2} - |A(P, Q)|,$$

where  $|A(P, Q)|$  are the paths from  $A$  to  $B$  which pass through  $P$  **and**  $Q$ . There are  $\binom{6}{3} \cdot \binom{8}{3} \cdot \binom{8}{2}$ . The answer is thus

$$\binom{6}{3} \cdot \binom{16}{5} + \binom{14}{6} \cdot \binom{8}{2} - \binom{6}{3} \cdot \binom{8}{3} \cdot \binom{8}{2}.$$

3. (20 points) Let us assume that we have 35 presents and 6 people. These 35 presents will now be distributed to these 6 people, and we always assume that the order in which the presents are received does *not* matter.
- (a) (10 points) Suppose that the presents are *different*. Find the number of ways to distribute these 35 presents between these 6 people.

**Solution.** There are 35 presents, and thus 35 choices to be made. These choices are independent as there is no constraint. There are 6 outcomes for each choice, and thus there are  $6^{35}$  ways to distribute the presents.  $\square$

- (b) (10 points) Suppose that the presents are *equal*. Find the number of ways to distribute these 35 presents between these 6 people.

**Solution.** This is equivalent to distributing  $35 + 6$  identical presents to 6 people such that each person gets at least one present. The answer is thus  $\binom{35+6-1}{6-1}$  ways. Alternatively, we can directly invoke Theorem 3.4.2 in the book and we also get  $\binom{35+6-1}{6-1}$ .  $\square$

4. (20 points) Solve the following two questions.

- (a) (10 points) Find the number of positive integers  $n \in \mathbb{N}$ , such that  $1 \leq n \leq 1000$ , which are **not** divisible by either 4, 10 or 15.

**Solution.** Let  $A_i$  be the number of positive integers  $n \in \mathbb{N}$ , such that  $1 \leq n \leq 1000$  divisible by  $i$ . The answer must then be

$$1000 - |A_4 \cup A_{10} \cup A_{15}|.$$

We apply the Inclusion-Exclusion Principle to compute  $|A_4 \cup A_{10} \cup A_{15}|$ . We have

$$\begin{aligned} |A_4 \cup A_{10} \cup A_{15}| &= |A_4| + |A_{10}| + |A_{15}| - (|A_4 \cap A_{10}| + |A_4 \cap A_{15}| + |A_{10} \cap A_{15}|) + |A_4 \cap A_{10} \cap A_{15}| = \\ &= \left\lfloor \frac{1000}{4} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left( \left\lfloor \frac{1000}{\text{lcm}(4, 10)} \right\rfloor + \left\lfloor \frac{1000}{\text{lcm}(4, 15)} \right\rfloor + \left\lfloor \frac{1000}{\text{lcm}(10, 15)} \right\rfloor \right) \\ &\quad + \left\lfloor \frac{1000}{\text{lcm}(4, 10, 15)} \right\rfloor. \end{aligned}$$

- (b) (10 points) How many permutations of 5 elements do **not** fix any element ?

**Solution.** Let  $A_i$  be the number of permutations of 5 elements which fix  $i$ . Since there are  $5!$  permutations, the answer will be

$$5! - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup |.$$

As above, we apply again the Inclusion-Exclusion Principle to compute  $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup |$ . Notice that all 5 sets  $A_i$  have cardinality  $4!$ , all the  $\binom{5}{2}$  double intersections have cardinality  $3!$ , all the  $\binom{5}{3}$  triple intersections have cardinality  $2!$ , all the  $\binom{5}{4}$  quadruple intersections have cardinality  $1!$  and the unique intersection  $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap |$  has cardinality 1. Thus we have

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup | = \binom{5}{1} \cdot 4! - \binom{5}{2} \cdot 3! + \binom{5}{3} \cdot 2! - \binom{5}{4} \cdot 1! + \binom{5}{5} \cdot 0!,$$

and the final answer is

$$5! - \left( \binom{5}{1} \cdot 4! - \binom{5}{2} \cdot 3! + \binom{5}{3} \cdot 2! - \binom{5}{4} \cdot 1! + \binom{5}{5} \cdot 0! \right).$$

5. (20 points) Let us have a French deck of 52 cards, containing 4 suits with the 13 values

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}$$

in each of the four suits. We are dealt five cards from the deck, and these five cards constitute our hand. The order in which we hold the cards is immaterial. Solve the following two questions.

- (a) (10 points) A hand of five cards is said to be a *flush* if it contains five cards with the same suit. Find the probability of drawing a flush.

**Solution.** Let us first choose the suit, which gives 4 choices, and then five cards in that suit, which gives  $\binom{13}{5}$  choices. There are thus  $4 \cdot \binom{13}{5}$  hands with a flush, and thus the probability is

$$\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}.$$

- (b) (10 points) A hand of five cards is said to have *two pairs* if it contains *exactly* two pairs of cards with the same value, the values of these pairs are *not* equal and the remaining card has a different value than the rest<sup>1</sup>. For example, 2 *threes*, 2 *Queens* and 1 King is a hand with two pairs, whereas 4 *Queens* and 1 King is not. Compute the probability of having a hand with two pairs.

**Solution.** First, choose the two values for the pair, which gives  $\binom{13}{2}$ , then choose the two suits in each of the pairs, which gives  $\binom{4}{2}$  in each choice, and thus  $\binom{4}{2}^2$  for both. We are left with the choice of the remaining fifth card, there are 11 possible values for it and 4 possible suits. In consequence, there are

$$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4,$$

hands with *two pairs*, and thus the probability is

$$\frac{\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4}{\binom{52}{5}}.$$

---

<sup>1</sup>In particular, there are no *three* cards with the same value in a hand with *two pairs*.