

University of California Davis
Combinatorics MAT 145

Name (Print): _____
Student ID (Print): _____

Sample Final Examination
Time Limit: 120 Minutes

March 22 2019

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- (A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.
- (B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.
- (C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- (D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	40	
Total:	120	

Do not write in the table to the right.

1. (20 points) (**Graph Profiling**) Consider the graph $G = (V, E)$ in Figure 1.

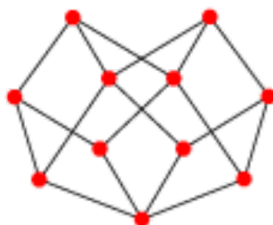


Figure 1: The graph for Problem 1.

- (a) (5 points) Show that the graph G is planar.
- (b) (10 points) Show that G is bipartite but it does *not* admits a perfect matching.
- (c) (5 points) Show that the graph G does *not* admits a Hamiltonian cycle.

2. (20 points) (**Prüfer Correspondence**) Solve the following two parts.

- (a) (10 points) Find the Prüfer Code associated to the three trees in Figure 2. The trees are denoted by T_1, T_2 and T_3 from left to right.

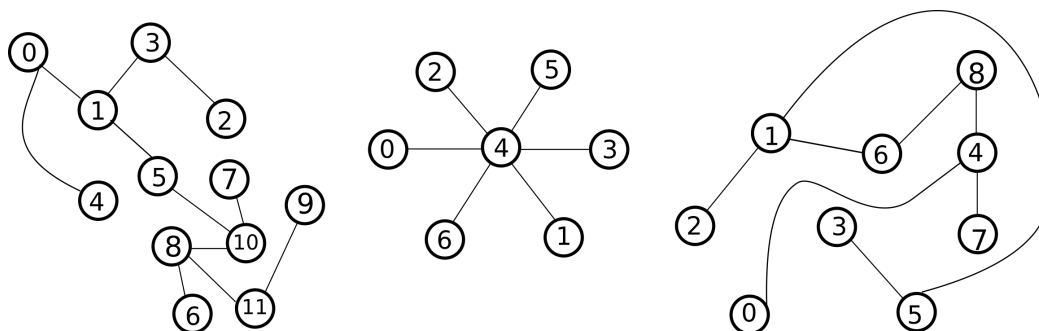


Figure 2: The trees T_1, T_2, T_3 for Problem 2.(a).

- (b) (10 points) Draw the labeled trees associated to the following three Prüfer codes:

$$\{0, 0, 4\}, \quad \{5, 6, 2, 3, 1\}, \quad \{0, 1, 2, 7, 4, 5\}.$$

3. (20 points) (**Perfect Matchings**) Solve the following two parts.
- (a) (10 points) Find two different perfect matchings for each of the three graphs in Figure 3. Draw the first perfect matching in the graph itself, and draw the second perfect matching right below.

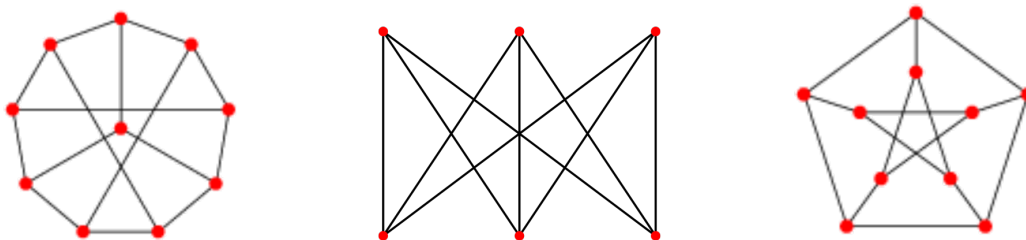


Figure 3: The three graphs for Problem 3.(a).

- (b) (10 points) Let $G = (V, E)$ be a connected bipartite graph with vertex bipartition $V = A \cup B$. Suppose that $\deg(x) \geq \deg(y)$ for all vertices x, y with $x \in A, y \in B$. Show that G admits a perfect matching.

4. (20 points) (**Planarity**) Solve the following two questions.

(a) (10 points) Let $T = (V, E)$ be a tree, show that T is planar.

(b) (10 points) Prove that the graph K_5 depicted in Figure 4 is *not* planar.

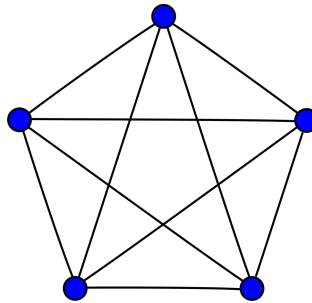


Figure 4: The graph K_5 for Problem 4.(b).

5. (20 points) (**Chromatic Properties**)

- (a) (10 points) Show that the chromatic polynomial of the cycle graph C_n in n vertices is given by the polynomial

$$\pi_{C_n} = (x - 1)^n + (-1)^n(x - 1).$$

These graphs C_n are depicted in Figure 5 for $n = 3, 4$ and 5 .

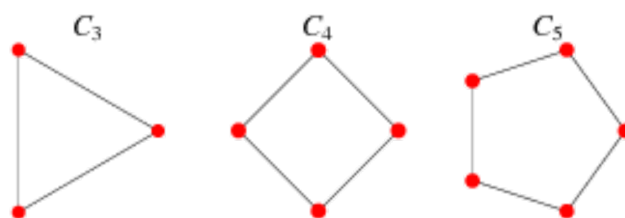


Figure 5: Cycle graphs C_n for $n = 3, 4, 5$.

- (b) (10 points) Find the number of 12 colorings of the graph K_7 , the complete graph in 7 vertices. The graph is depicted in Figure 6.

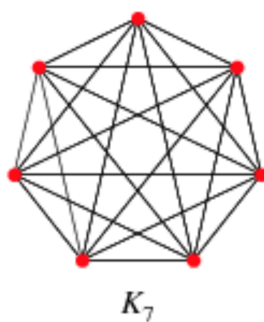


Figure 6: The complete graph K_7 .