**University of California Davis** **Name (Print):** ____________________
**Combinatorics MAT 145** **Student ID (Print):** ____________________

**Sample Final Examination II** **March 22 2019**
**Time Limit: 120 Minutes**

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This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 40 | |
| Total: | 120 | |

1. (20 points) (**Graph Profiling**) Consider the graph $G = (V, E)$ in Figure 1.

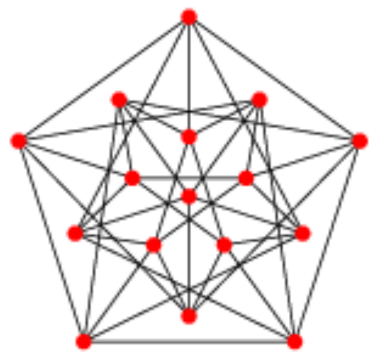

Figure 1: The graph for Problem 1.

(a) (5 points) Show that $G$ does *not* admits an Euler cycle.

(b) (5 points) Prove that there exists a Hamiltonian cycle.

(c) (5 points) Prove that $\chi(G) \geq 3$, i.e. $G$ is *not* bipartite.

(d) (5 points) Let $T$ be a spanning graph for $G$. How many edges does $T$ have ?

2. (20 points) (**Trees and Cayley's Theorem**) Let $B_n$ be the $(n, n)$-dumbbell graph, obtained by joining two disjoint complete $K_n$ graphs with one edge, as depicted in Figure 2 for the cases $n = 4, 5, 6, 7$.

   (a) (10 points) Show that the number of spanning trees of $B_n$ is $n^{2n-4}$.

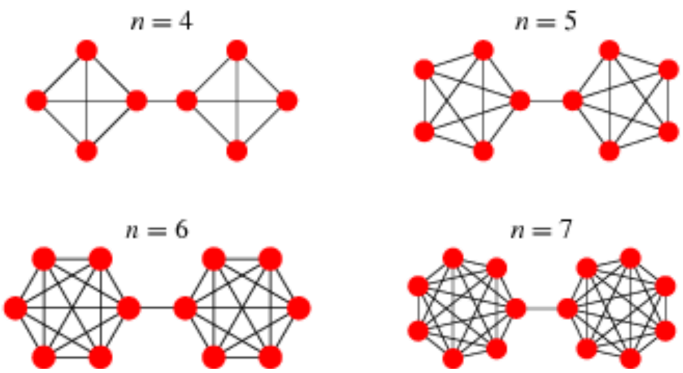


Figure 2: The $(n, n)$-dumbbell graphs for $n = 4, 5, 6, 7$.

   (b) (10 points) Let $T_n$ be the number of *unlabeled* trees in $n$ vertices. Show that this number satisfies the inequality

$$n^{n-2} \leq n!T_n.$$

3. (20 points) (**Perfect Matchings**) Solve the following two parts.

   (a) (5 points) Show that a tree $T$ has at most one perfect matching.

   (b) (10 points) Prove that a bipartite graph $G$ such that all vertices have the same degree admits a perfect matching.

   (c) (5 points) Construct a graph $G = (V, E)$ in which all vertices have the same degree but $G$ does *not* admit a perfect matching.

4. (20 points) (**Planarity And Colorings**) Consider the graph $G = (V, E)$ in Figure 3.

   (a) (10 points) Show that $G$ is *not* planar.

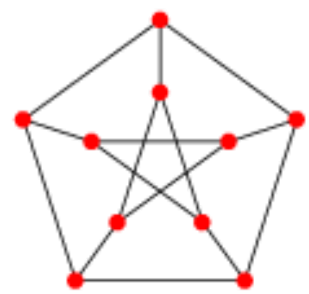

Figure 3: The Petersen Graph for Problem 4.

   (b) (10 points) Find the chromatic number $\chi(G)$ of $G$.

5. (20 points) (**Chromatic Properties**)

   (a) (10 points) Show that the chromatic polynomial for the complete bipartite graph $K_{2,3}$ is given by the polynomial

$$\pi_{K_{2,3}} = x(x-1)^3 + x(x-1)(x-2)^3.$$

   These graphs $C_n$ are depicted in Figure 4 for $n = 3, 4$ and 5.

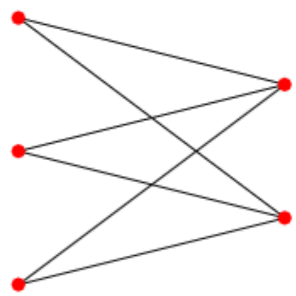

Figure 4: The complete bipartite graph $K_{2,3}$.

   (b) (10 points) Let $G$ be a graph with chromatic polynomial

$$\pi_G(x) = x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x.$$

   Show that $G$ is non-planar.