University of California Davis
Combinatorics MAT 145

Name (Print): ____________________
Student ID (Print): ____________________

Sample Final Examination II

March 22 2019

Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	40	
Total:	120	

1. (20 points) (**Graph Profiling**) Consider the graph $G = (V, E)$ in Figure 1.

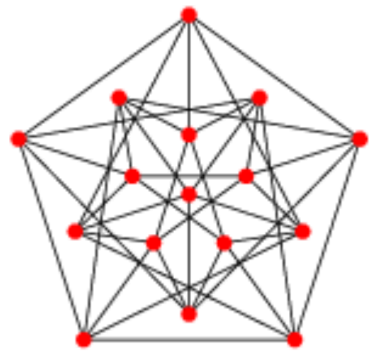

Figure 1: The graph for Problem 1.

(a) (5 points) Show that G does *not* admits an Euler cycle.

Solution. We have an Euler cycle if and only if every vertex has even degree. The vertex in the very middle (and many others) has degree 5. □

(b) (5 points) Prove that there exists a Hamiltonian cycle.

Solution. A Hamiltonian cycle can be found directly in the graph. □

(c) (5 points) Prove that $\chi(G) \geq 3$, i.e. G is *not* bipartite.

Solution. Note that the outer pentagon is an odd cycle, thus G is not bipartite. □

(d) (5 points) Let T be a spanning tree for G. How many edges does T have ?

Solution For trees $|E| = |V| - 1$ we know that G has 16 vertices so a spanning tree should have 15 edges. □

2. (20 points) (**Trees and Cayley's Theorem**) Let B_n be the (n, n)-dumbbell graph, obtained by joining two disjoint complete K_n graphs with one edge, as depicted in Figure 2 for the cases $n = 4, 5, 6, 7$.

   (a) (10 points) Show that the number of spanning trees of B_n is n^{2n-4}.

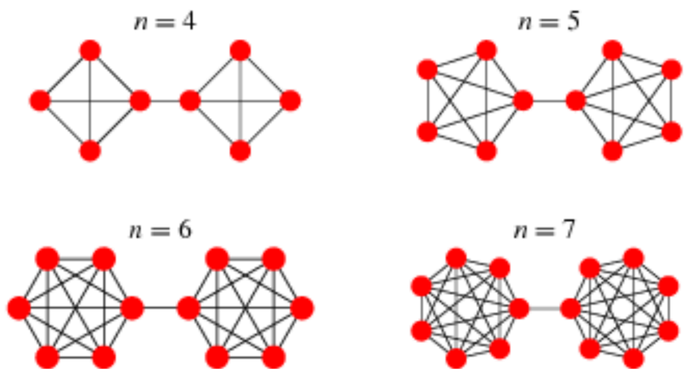

Figure 2: The (n, n)-dumbbell graphs for $n = 4, 5, 6, 7$.

Solution. Every spanning tree must contain the bridge edge. Now the rest of the tree must be comprised of 2 different copies of a spanning tree for K_n starting at the ends of the bridge edge. Note that in an unlabeled tree *any* node can be taken to be the root. In particular, we let the nodes of the bridge edge to be the roots of the left and right copies of K_n. Now recall that the number of spanning trees of K_n is n^{n-2} (Problem Set 5). Multiplying we get the total number of spanning trees of B_n is n^{2n-4}. □

   (b) (10 points) Let T_n be the number of *unlabeled* trees in n vertices. Show that this number satisfies the inequality

$$n^{n-2} \leq n! T_n.$$

Solution. The left hand side is counting the number of spanning trees of a *labeled* graph on n vertices. Clearly if we start with any unlabeled tree on n nodes and then assign labels to the nodes in $n!$ ways (which the right hand side is counting) we must be getting at least as many different labeled trees as spanning trees of a labeled graph on n vertices. □

3. (20 points) (**Perfect Matchings**) Solve the following two parts.

   (a) (5 points) Show that a tree T has at most one perfect matching.

   Solution. If the order of the tree is odd there are no perfect matchings so let's assume it is even ($2n$ nodes). We know that every tree has at least one leaf. We will construct the perfect matching iteratively. If there is to be a perfect matching every leaf must be matched to their neighbor. Start by taking it a leaf and matching it to its neighbor. Then delete the rest of the edges coming out of that neighbor. Now we are left with a matched pair of nodes and a smaller tree on $2n - 2$ nodes. This tree again has at least one leaf so we reiterate the procedure. Note that at every step the leaves can be matched to only one other vertex and that every node will eventually become a leaf as we prune the tree. Therefore there is exactly one perfect matching. □

   (b) (10 points) Prove that a bipartite graph G such that all vertices have the same degree admits a perfect matching.

   Solution. Let k be such degree. The total number of outgoing edges from A has to match those incoming to B. This yields the relationship $k|A| = k|B|$ which shows $|A| = |B|$. Now let $S \subset A$ where $|S| = m$. Similarly, the number of outgoing edges from S has to match the incoming edges to $N(S)$. Suppose by contradiction that $|N(S)| < m$. We have $m \cdot k$ outgoing edges from A but $|N(S)| \cdot k < m \cdot k$ incoming edges into $N(S)$ Thus the two conditions for Hall's theorem are fulfilled and we have a perfect matching. □

   (c) (5 points) Construct a graph $G = (V, E)$ in which all vertices have the same degree but G does *not* admit a perfect matching.

   Solution. Consider any cyclic graph C_n where n is odd. Then all vertices have degree 2 but there is no perfect matching because there is an odd number of vertices. □

4. (20 points) (**Planarity And Colorings**) Consider the graph $G = (V, E)$ in Figure 3.

   (a) (10 points) Show that G is *not* planar.

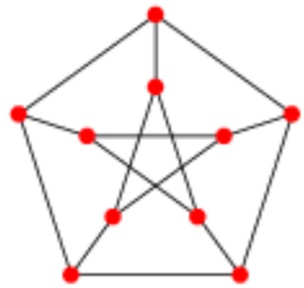

Figure 3: The Petersen Graph for Problem 4.

Solution. Recall that if a graph is planar so are all its subgraphs. Petersen's graph contains $K_{3,3}$ which is not planar. (see Problem Set 6, question 8). Alternatively, Euler's formula yields that a planar embedding of G will have exactly $f = 2-v+3 = 2-10+15 = 7$ faces. Since G has no 3 or 4-cycles, it must be that for such a planar embedding $5f \leq 2e$. This is a contradiction, since $5 \cdot 35 \leq 2 \cdot 10$ does not hold. □

   (b) (10 points) Find the chromatic number $\chi(G)$ of G.

Solution. The chromatic number is at most 3 because there exists a 3-coloring as shown below. On the other hand it has to be strictly greater than 2 because it is not bipartite;indeed, the outer pentagon is an odd cylce. Hence $\chi(G) = 3$. □

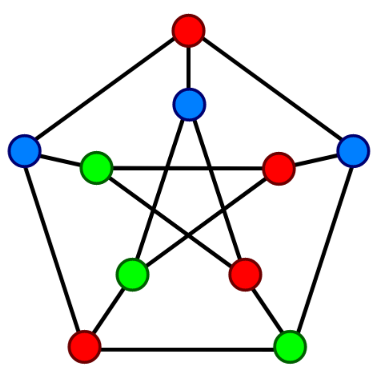

Figure 4: A 3-coloring of the Petersen graph.

5. (20 points) (**Chromatic Properties**)

(a) (10 points) Show that the chromatic polynomial for the complete bipartite graph $K_{2,3}$ is given by the polynomial

$$\pi_{K_{2,3}} = x(x-1)^3 + x(x-1)(x-2)^3.$$

These graphs C_n are depicted in Figure 5 for $n = 3, 4$ and 5.

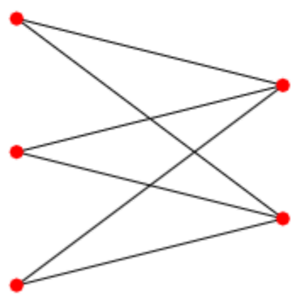

Figure 5: The complete bipartite graph $K_{2,3}$.

Solution. The three nodes on the left must have colors different to the two on the right. Now let's split into two cases. Since there are no edges connecting nodes within each set of the bipartition the two nodes on the right may or may not have the same color. These are the two cases we distinguish and correspond to each of summand in the resulting chromatic polynomial. In the first case, the two nodes on the right have the same color, and there are x choices for such color. This leaves $(x-1)$ possible colors for each of the 3 nodes on the left. This accounts for the $x(x-1)^3$ term. In the second case, where the two nodes on the right do have different colors, we can pick the color for the first one in x ways and that for the second one in $(x-1)$ ways; this leaves $(x-2)$ possibilities for each of the 3 nodes on the left. This accounts for the second $x(x-1)(x-2)^3$ summand. Summing over these two alternate possibilities we have

$$\pi_{K_{2,3}} = x(x-1)^3 + x(x-1)(x-2)^3.$$

□

(b) (10 points) Let G be a graph with chromatic polynomial

$$\pi_G(x) = x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x.$$

Show that G is non-planar.

Solution Recall that the degree of the polynomial is the number of nodes and that the coefficient of x^{n-1} is $-e$. Then the inequality $e \leq 3n - 6$ is not satisfied so G is non-planar. Alternatively, note that $\pi_G(4) = 0$ and thus G does not admit a 4-coloring. Since any planar graph admits a 4-coloring, it must be that G is non-planar. □