**University of California Davis**
**Combinatorics MAT 145**

**Name (Print):** ____________________
**Student ID (Print):** ____________________

**Sample Final Examination**
**Time Limit: 120 Minutes**

**March 22 2019**

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This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 40 | |
| Total: | 120 | |

1. (20 points) (**Graph Profiling**) Consider the graph $G = (V, E)$ in Figure 1.

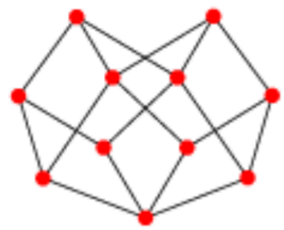


Figure 1: The graph for Problem 1.

(a) (5 points) Show that the graph $G$ is planar.

**Solution.** A planar embedding of $G$ is depicted in Figure 2. □

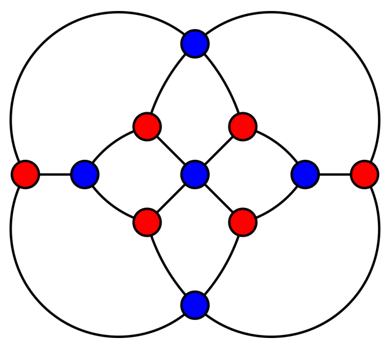


Figure 2: A bipartite planar embedding for the graph $G$.

(b) (10 points) Show that $G$ is bipartite but it does *not* admits a perfect matching.

**Solution.** A bipartition of the vertex set if presented in Figure 2. By Hall's Theorem, it is a necessary condition for the existence of a perfect matching of a bipartite graph that the bipartition $V = A \cup B$ of the vertex set $V$ consists of the two sets of the *same* size, thus $|A| = |B|$. This is impossible since the number of vertices is 11, which is odd, and thus it must be that $|A| \neq |B|$, violating the condition in Hall's Theorem. □

(c) (5 points) Show that the graph $G$ does *not* admits a Hamiltonian cycle.

**Solution.** A Hamiltonian cycle $\gamma$ in a bipartite graph, with $V = A \cup B$, must alternate between vertices in $A$ and $B$. Thus, for such Hamiltonian cycle $\gamma$ to exist we must have $|A| = |B|$, which, as above, cannot happen. Hence, no such $\gamma$ exists. □

2. (20 points) (**Prüfer Correspondence**) Solve the following two parts.

   (a) (10 points) Find the Prüfer Code associated to the three trees in Figure 3. The trees are denoted by $T_1, T_2$ and $T_3$ from left to right.

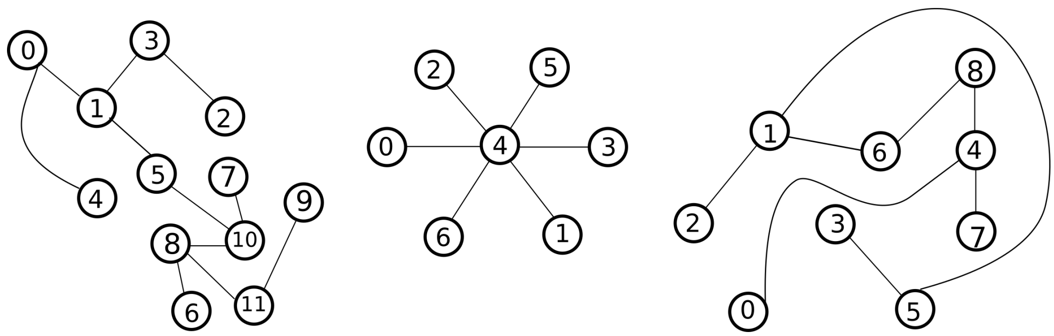



Figure 3: The trees $T_1, T_2, T_3$ for Problem 2.(a).

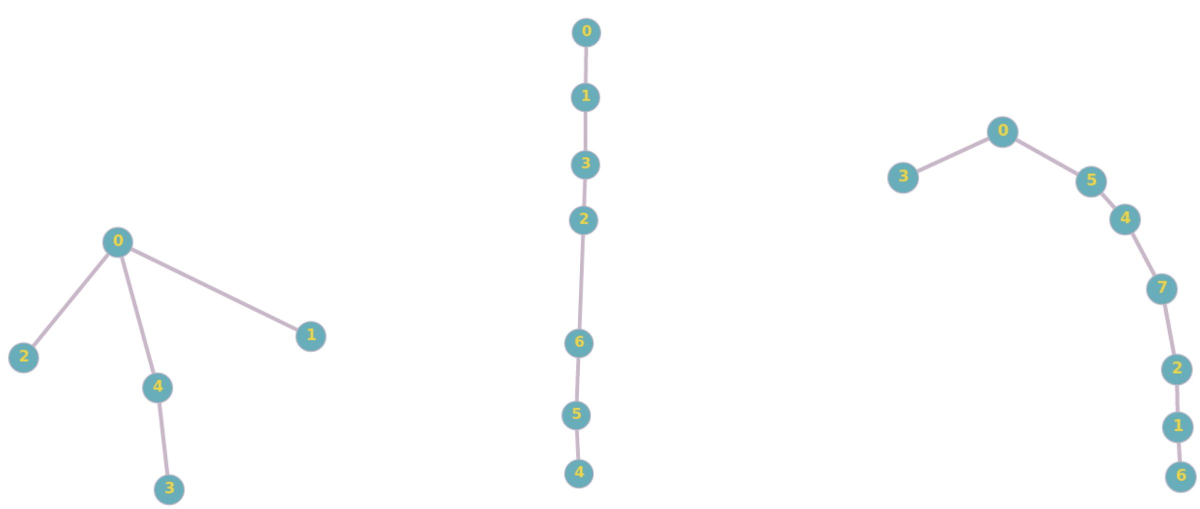
**Solution.**

$T_1$ : $\{3, 1, 0, 8, 10, 11, 8, 10, \}$
$T_2$ : $\{4, 4, 4, 4, 4\}$
$T_3$ : $\{1, 5, 1, 6, 8, 4, 4\}$

   (b) (10 points) Draw the labeled trees associated to the following three Prüfer codes:

$$\{0, 0, 4\}, \quad \{5, 6, 2, 3, 1\}, \quad \{0, 1, 2, 7, 4, 5\}.$$

**Solution.** Below are the respective graphs. In the solutions to Problem Set 5 the algorithm for converting a Prufer coede into a tree is given. Note that if we insist that 0 is always a root when we pick the smallest available label in the alphabet we exclude zero; that is we pick the smallest label other than 0. □



3. (20 points) (**Perfect Matchings**) Solve the following two parts.

(a) (10 points) Find two different perfect matchings for each of the three graphs in Figure 4. Draw the first perfect matching in the graph itself, and draw the second perfect matching right below.

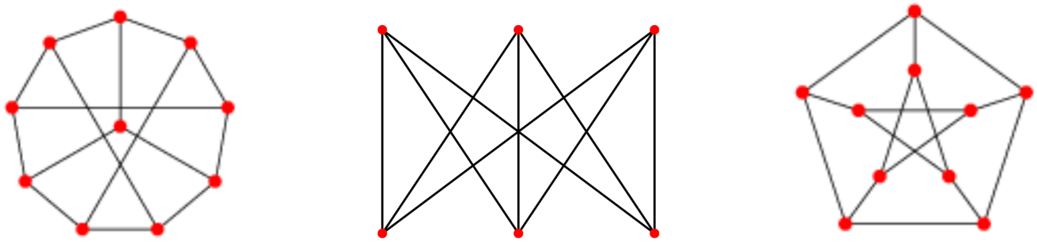


Figure 4: The three graphs for Problem 3.(a).

**Solution.** For the first graph take the center node and match it to one of its neighbors. Then the rest of the 8 nodes are all in the outer cycle and we can match them to each other by deleting every other vertex. Another perfect matching can be constructed following exactly these procedure but matching the center vertex to a different neighbor instead.

This is $K_{3,3}$ so there are in fact 3! matchings. One is to match every node on the top to the one opposite to it on the bottom. Another possibility is to match the first on top to the second on the bottom, second on top to the third on the bottom and the third on the top to the first on the bottom.

The easiest matching is perhaps to simply use the edges connecting the outer pentagon with the inner star. Another possibility is to take two nonconsecutive edges in the outer pentagon. Now there is one node remaining in the pentagon so we match it to the node it is adjacent to in the inner star. Finally we match the 4 remaining nodes in the inner star pairwise. $\square$

(b) (10 points) Let $G = (V, E)$ be a connected bipartite graph with vertex bipartition $V = A \cup B$. Suppose that $\deg(x) \geq \deg(y)$ for all vertices $x, y$ with $x \in A$, $y \in B$. Show that $G$ admits a perfect matching.

**Solution.** By Hall's theorem it suffices to show that for all $S \subset A$ then $|N(S)| \geq |S|$. Let $k := \max_{\{y \in B\}} \deg(y)$, then by the assumption the number of edges coming out have $S$ is at least $k|S|$ which has to match the number of edges connecting to $N(S)$. However if $|N(S)| < |S|$ then $\{\#\text{edges connecting to } N(S)\} \leq k|N(S)| < k|S|$, a contradiction. $\square$

4. (20 points) (**Planarity**) Solve the following two questions.

   (a) (10 points) Let $T = (V, E)$ be a tree, show that $T$ is planar.

   **Solution.** We proceed by induction. For $n = 2$ then $T$ is clearly planar. Now assume trees with $|V| = n$ are planar. Every tree with $|V| = n+1$ can be constructed by adding a leaf. Let's take the node adjacent to this leaf to be the root . Then we place the leaf on one side (half-plane) of a line though the root and the rest of the tree we can embed it on the other half plane by inductive assumption. $\square$

   (b) (10 points) Prove that the graph $K_5$ depicted in Figure 5 is *not* planar.

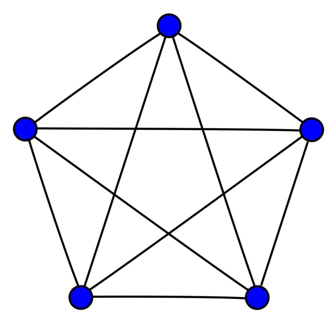


Figure 5: The graph $K_5$ for Problem 4.(b).

   **Solution** Recall that planar graphs satisfy $e \leq 3v - 6$. Since $K_5$ is complete we have as many edges as pairs of vertices so $e = \binom{5}{2} = 10 > 9 = 3 \cdot 5 - 6$. $\square$

5. (20 points) (**Chromatic Properties**)

(a) (10 points) Show that the chromatic polynomial of the cycle graph $C_n$ in $n$ vertices is given by the polynomial

$$\pi_{C_n} = (x-1)^n + (-1)^n(x-1).$$

These graphs $C_n$ are depicted in Figure 6 for $n = 3, 4$ and 5.

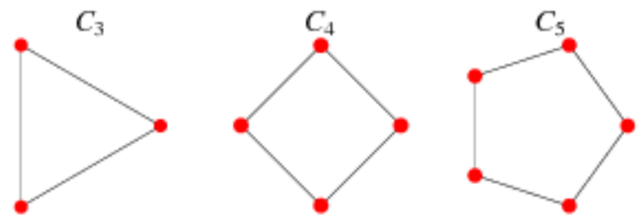



Figure 6: Cycle graphs $C_n$ for $n = 3, 4, 5$.

**Solution.** We will prove this by induction. For the base case $n = 3$ clearly all vertices must have different colors so

$$\pi_{C_3} = x(x-1)(x-2) = (x-1)(x^2-2x) = (x-1)((x-1)^2-1) = (x-1)^3 - (x-1).$$

For the inductive step, assume the formula holds for $C_n$ then by the deletion-contraction formula:

$$\pi_{C_{n+1}}(x) := P(C_{n+1}, x) = P(C_{n+1} - e, x) - P(C_{n+1} \cdot e, k) = \pi_{S_{n+1}}(x) - \pi_{C_n}(x)$$

Here $G - e$ is the graph obtained by deleting an edge $e$ in $C_{n+1}$ which in this case is the straight line graph $S_{n+1}(x)$. On the other hand $C_{n+1} \cdot e$ is the graph obtained by contracting $e$ in $G$; that is $C_n$.( Here $\pi$ and $P$ both represent the chromatic polynomial. The notation change was to emphasize the deletion and contraction). Note that the last graph is a tree so $\pi_{S_{n+1}}(x) = x(x-1)^n$. Thus using the inductive hypothesis for $\pi_{C_n}(x)$:

$$\pi_{C_{n+1}}(x) = x(x-1)^n - ((x-1)^n + (-1)^n(x-1)) = (x-1)^{n+1} + (-1)^{n+1}(x-1).$$

□

(b) (10 points) Find the number of 12 colorings of the graph $K_7$, the complete graph in 7 vertices. The graph is depicted in Figure 7.

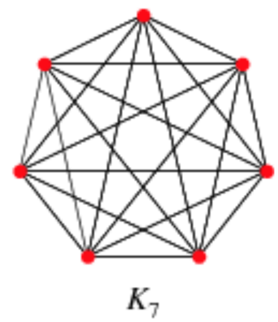



Figure 7: The complete graph $K_7$.

**Solution.** The number of 12 colorings of $K_7$ is given $\pi_{K_7}(12)$. From the Problem Set 7 we know that $\pi_{K_7}(x) = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$. Evaulating at 12 yields 3991680 colorings. $\square$