University of California Davis **Name (Print):** ____________________
Combinatorics MAT 145 **Student ID (Print):** ____________________

Sample Midterm Examination **February 8 2019**
Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Prove the following two statements.

   (a) (10 points) Prove that for every $n \in \mathbb{N}$

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

   (b) (10 points) Show that for every $n \in \mathbb{N}$

$$\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^k\binom{n}{k} + \ldots + 2^n\binom{n}{n} = 3^n.$$

2. (20 points) Consider a 5×6 rectangular grid as depicted in Figure 1. A staircase walk is a path in the grid which moves only *right* or *up*.

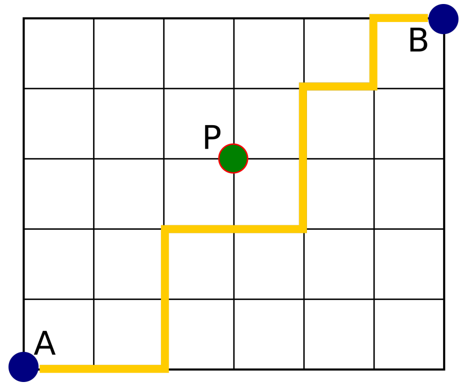

Figure 1: The 5×6 grid and a valid staircase walk from A to B avoiding P.

(a) (10 points) Find the number of staircase walks from A to B.

(b) (10 points) How many staircase walks from A to B avoid the point P ?

3. (20 points) Solve the following two questions.

   (a) (10 points) Compute the number of positive integers $n \in \mathbb{N}$ such that $1 \leq n \leq 10000$, and which are **not** divisible by $6, 7, 9$.

   (b) (10 points) Prove that in any group of six people there are either three people who all know each other, or three people who are mutual strangers.

4. (20 points) Consider a French deck of 52 cards, containing 4 suits with 13 values

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}$$

in each of the four suits. We are dealt five cards from the deck. These five cards constitute our hand.

(a) (10 points) A hand of five cards is said to be a *trio* if it contains *exactly* three cards with the same value, and the remaining two cards have distinct values amongst them. Show that the probability of having a trio is approximately 0.021128.

(b) (10 points) A hand of five cards is said to be a *full house* if it contains *exactly* three cards with the same value, and the remaining two cards have the same value. Compute the probability of having a full house.

5. (20 points) Let us consider a die with six faces, each face with a value in $\{1, 2, 3, 4, 5, 6\}$. Roll the die four consecutive times. Compute the probability that we will see a 3 in at least one of these four rolls.