University of California Davis **Name (Print):** ____________________
Combinatorics MAT 145 **Student ID (Print):** ____________________

Sample II Midterm Examination **February 8 2019**
Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Prove the following two statements.

   (a) (10 points) Let $n \in \mathbb{N}$ be an even number, and $n \geq 2$.
   Prove the following inequality:

$$\frac{2^n}{n+1} < \binom{n}{n/2}.$$

   (b) (10 points) Show that for every $n \in \mathbb{N}$

$$\sum_{k=1}^{n} k(n-k+1) = \binom{n+2}{3}.$$

2. (20 points) Suppose we have 15 identical 1\$ dollar bills.

   (a) (10 points) Find the number of ways to distribute these 15 one dollar bills between 3 people such that the first person gets 3\$, the second person get 4\$ and the third person gets 8\$.

   (b) (10 points) How many ways are there to distribute these 15 one dollar bills between 7 people such that each person gets *at least* one dollar bill ?

3. (20 points) Let $n \in \mathbb{N}$ be a natural number and $P = \{p_1, p_2, \ldots, p_s\}$ the set of all distinct primes which divide n.

   (a) (5 points) Show that for a prime $p_1 \in P$, the number of natural numbers $k \in \mathbb{N}$, $k \leq n$, that share a factor of p_1 with n is $\frac{n}{p_1}$.

   (b) (5 points) Show that for two primes $p_1, p_2 \in P$, the number of natural numbers $k \in \mathbb{N}$, $k \leq n$, that share factors of p_1 and p_2 with n is $\frac{n}{p_1 \cdot p_2}$.

   (c) (10 points) Show that the number of natural numbers $k \in \mathbb{N}$, $k \leq n$, which share *no* prime factor with n equals

$$n \cdot \prod_{p \in P} \left(1 - \frac{1}{p}\right).$$

   Hint: You might want to use the Inclusion-Exclusion Principle, considering subsets depending on the number of common prime factors, in line as in Parts (a) and (b).

4. (20 points) Consider a 3×11 rectangular grid as depicted in Figure 1, formed by 33 tiles of area $1m^2$. A staircase walk is a path in the grid which moves only *right* or *up*.

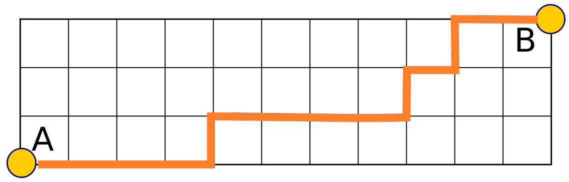

Figure 1: The $3{\times}11$ grid and a staircase walk from A to B with area $12m^2$.

(a) (10 points) How many staircase walks are there from A to B which exactly start by going to the right two times ?

(b) (10 points) The area of a path is the number of tiles underneath the path in m^2 units. What is the probability that a staircase walk from A to B has area $12m^2$?

Hint: You might want to use following two identities

$$(q^{14}-1) = (q^2-1)\cdot(q^{12}+q^{10}+q^8+q^6+q^4+q^2+1), \quad (q^{12}-1) = (q^3-1)\cdot(q^9+q^6+q^3+q+1).$$

5. (20 points) Consider a configuration of five ordered cards with distinct values $\{1, 2, 3, 4, 5\}$. The cost of a configuration is defined to be the minimal number of switches of two *consecutive* positions needed to achieve the standard order $1, 2, 3, 4$ and 5. For instance $1, 3, 2, 5, 4$ would have a cost of 2$, and $3, 1, 2, 5, 4$ a cost of 3$.

   Suppose that the cards are distributed with a uniform random probability. Show that the probability of having a configuration of cost 8$ is 0.075.