University of California Davis **Name (Print):** ______________________

Combinatorics MAT 145 **Student ID (Print):** ______________________

Sample II Midterm Examination **February 8 2019**

Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Prove the following two statements.

   (a) (10 points) Let $n \in \mathbb{N}$ be an even number, and $n \geq 2$.
   Prove the following inequality:

$$\frac{2^n}{n+1} < \binom{n}{n/2}.$$

Solution. The base case is $n = 2$ and indeed we have $\frac{4}{3} = \frac{2^2}{2+1} < \binom{2}{1} = 2$. For the inductive step we have:

$$\frac{2^{n+2}}{(n+2)+1} = \frac{4(n+1)}{(n+3)} \frac{2^n}{(n+1)} < \frac{(n+2)(n+1)}{(\frac{n}{2}+1)^2} \binom{n}{n/2} = \binom{n+2}{n/2+1}.$$

Here the inequality follows by inductive hypothesis and the observation that:

$$n^2 + 5n + 6 \geq n^2 + 4n + 4 \implies \frac{(n+2)}{(n/2+1)^2} \geq \frac{4}{(n+3)}.$$

   (b) (10 points) Show that for every $n \in \mathbb{N}$

$$\sum_{k=1}^{n} k(n-k+1) = \binom{n+2}{3}.$$

Solution. The right hand side counts the number of ways to pick 3 integers out of the $\{0, 1, 2 \ldots n+1\}$. As for the right hand side each term gives you the number of ways to choose 3 integers out of $\{0, 1, 2 \ldots n+1\}$ conditioned on the middle one being equal to k. Summing over all possibilities we get the desired result.

2. (20 points) Solve the following two problems.

(a) (10 points) Suppose we have 15 **distinct**[1] 1\$ dollar bills. Find the number of ways to distribute these 15 one dollar bills between 3 people such that the first person gets 3\$, the second person get 4\$ and the third person gets 8\$.

Solution. In other words we need to count the ways to divide a set of cardinality 15 into subsets of sizes 3, 4 and 8. This is given by the multinomial coefficient formula:

$$\frac{15!}{3!4!8!}.$$

(b) (10 points) How many ways are there to distribute *identical* 15 one dollar bills between 7 people such that each person gets *at least* one dollar bill ?

Solution. Since the bills are identical, we have $\binom{15-1}{7-1} = \binom{14}{6}$ choices, as we can align the bills and then we can insert the 7 people in the 14 spaces between the 15 bills, where the first person starts by being fixed at the leftmost space. $\square$

[1] In the initial version the Problem was for identical bills. The usual is much simpler in that case.

3. (20 points) Let $n \in \mathbb{N}$ be a natural number and $P = \{p_1, p_2, \ldots, p_s\}$ the set of all distinct primes which divide n.

   (a) (5 points) Show that for a prime $p_1 \in P$, the number of natural numbers $k \in \mathbb{N}$, $k \leq n$, that share a factor of p_1 with n is $\frac{n}{p_1}$.

   Solution. We can verify this with our bare hands; they are: $\{p_1, 2p_1, \ldots \frac{n}{p_1} p_1\}$.

   (b) (5 points) Show that for two primes $p_1, p_2 \in P$, the number of natural numbers $k \in \mathbb{N}$, $k \leq n$, that share factors of p_1 and p_2 with n is $\frac{n}{p_1 \cdot p_2}$.

   Solution. Likewise, there are the multiples of $p_1 p_2$ smaller than n. There are $\frac{n}{p_1 p_2}$ of them namely $\{p_1 p_2, 2p_1 p_2, \ldots \frac{n}{p_1 p_2} p_1 p_2\}$.

   (c) (10 points) Show that the number of natural numbers $k \in \mathbb{N}$, $k \leq n$, which share *no* prime factor with n equals

$$n \cdot \prod_{p \in P} \left(1 - \frac{1}{p}\right).$$

   Hint: You might want to use the Inclusion-Exclusion Principle, considering subsets depending on the number of common prime factors, in line as in Parts (a) and (b).

   Solution. This quantity is interesting and is called is called Euler's totient function $\varphi(n)$. Let A_k be the set of naturals $k \leq n$ divisible by k-th prime p_k. Note that the cardinality we are interested in is $n - |\cup_{p_k \in P} A_k|$. We find this by inclusion exclusion and using the previous parts:

$$|\cup_{p_k \in P} A_k| = \sum_{p_k \in P} |A_k| - \sum_{p_k, p_j \in P} |A_k \cap A_j| + \ldots + (-1)^m |A_1 \cap A_2 \cap \ldots \cap A_m|$$

$$= n \left(\sum_{p_k \in P} \frac{1}{p_k} - \sum_{p_k, p_j \in P} \frac{1}{p_j p_k} + \ldots + (-1)^m \frac{1}{p_1 p_2 \ldots p_m} \right)$$

   Using the mutinomial coefficient expansion we have that:

$$n \left(1 - \sum_{p_k \in P} \frac{1}{p_k} + \sum_{p_k, p_j \in P} \frac{1}{p_j p_k} + \ldots + (-1)^{m+1} \frac{1}{p_1 p_2 \ldots p_m}\right) = n \prod_{p_k \in P} \left(1 - \frac{1}{p_k}\right).$$

4. (20 points) Consider a 3×11 rectangular grid as depicted in Figure 1, formed by 33 tiles of area $1m^2$. A staircase walk is a path in the grid which moves only *right* or *up*.

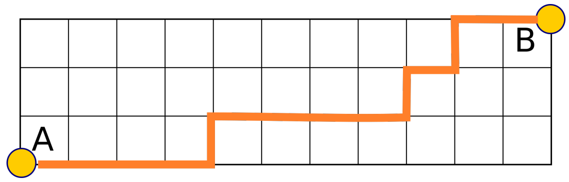

Figure 1: The $3{\times}11$ grid and a staircase walk from A to B with area $12m^2$.

(a) (10 points) How many staircase walks are there from A to B which exactly start by going to the right two times ?

Solution. We can only reach the grid point $(2,0)$ by making two moves to the right in the beginning. Therefore the total number of staircase walks on this grid that start by moving right twice is exactly the same as the number of staircase walks from $(2,0)$ to B namely $\binom{12}{3}$.In other words we need to choose 3 up moves out of a total of 12 remaining moves to make it to B (the rest are right moves).

(b) (10 points) The area of a path is the number of tiles underneath the path in m^2 units. What is the probability that a staircase walk from A to B has area $12m^2$?

Hint: You might want to use following two identities

$$(q^{14}-1) = (q^2-1)\cdot(q^{12}+q^{10}+q^8+q^6+q^4+q^2+1), \quad (q^{12}-1) = (q^3-1)\cdot(q^9+q^6+q^3+1).$$

Solution. The binomial coefficient

$$\binom{n+m}{m}_q = \prod_{i=0}^{m} \frac{1-q^{n+m-i}}{1-q^{i+1}}$$

can be interpreted as the polynomial in q whose q^k coefficient gives the number of *distinct* ways to fit k squares into an $m \times n$ rectangular grid. In the context of this problem distinct means that we can impose the condition that a given row must contain at most as many elements as the row underneath it. In this case we wish to find the number of distinct ways to fit 12 unit squares inside 3×11 grid. Using the formula with $n = 11$ and $m = 3$ we want to find the q^{12} coefficient of:

$$\frac{(1-q^{14})(1-q^{13})(1-q^{12})}{(1-q^3)(1-q^2)(1-q)} = (q^{12}+q^{10}+q^8+q^6+q^4+q^2+1)(q^9+q^6+q^3+q+1)(1+q+q^2+...q^{12})$$

Here we used the given identities and the fact that $\frac{1-1^{13}}{1-q} = 1 + q + q^2 ... + q^{11} + q^{12}$. We can brute force the counting of occurences of q^{12} in this product to find that the desired coefficient is 24. This is actually easier than it looks. Proceed case by case in the summands of the first product and for each case of $k = 12, 10, ...2, 1$ there are as many contributions to the coefficients as terms in $(q^9 + q^6 + q^3 + q + 1)$ with power less than or equal to $12 - k$.

5. (20 points) Consider a configuration of five ordered cards with distinct values $\{1, 2, 3, 4, 5\}$. The cost of a configuration is defined to be the minimal number of switches of two *consecutive* positions needed to achieve the standard order $1, 2, 3, 4$ and 5. For instance $1, 3, 2, 5, 4$ would have a cost of 2\$, and $3, 1, 2, 5, 4$ a cost of 3\$.

   Suppose that the cards are distributed with a uniform random probability. Show that the probability of having a configuration of cost 8\$ is 0.075.

   Solution. The first observation is that the cost of a permutation is precisely the number of pairs of numbers in it whose order is inverted for instance in $\{5, 1, 2, 3, 4\}$ the cost is 4 because each of the pairs $(5, 1)$,$(5, 1)$,$(5, 1)$,$(5, 1)$ is inverted order while all the pairs of numbers in 1234 are clearly in their natural order. Take a minute to convince yourself this is the case. The problem is then down to finding how many out the 120 permutations of 5 distinct values have 8 inverted pairs. Now clearly there are $\binom{5}{2} = 10$ pairs of 2 integers $\{1, 2, 3, 4, 5\}$. Then we could naively guess that there are simply $\binom{10}{8} = 45$ permutations with 8 inverted pairs. However this is overcounting. In fact, let's fix the positions of one of the inverted pairs. To arrange the other 3 numbers we still have 9 other relative position including 1 inversion to allocate. This gives us $\binom{9}{1}$ configurations with 8 inversions. Note that $9/120 = 0.075$. □