University of California Davis **Name (Print):** ____________________

Combinatorics MAT 145 **Student ID (Print):** ____________________

Sample Midterm Examination **February 8 2019**

Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Prove the following two statements.

   (a) (10 points) Prove that for every $n \in \mathbb{N}$

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Solution. The left hand side counts the number of 2-element subsets of a $(2n)$-element set X. Let us count the element of X differently. Consider X as a union of two copies of $\{1, 2, \ldots, n\}$, so that

$$X = \{1, 2, \ldots, n\} \cup \{1, 2, \ldots, n\}.$$

Then a pair of elements can come from the first copy, contributing $\binom{n}{2}$, from the second copy, contributing $\binom{n}{2}$, or each element from a different copy, giving n^2. These three cases contribute a total of

$$2\binom{n}{2} + n^2,$$

which is the right hand side, as requested. □

   (b) (10 points) Show that for every $n \in \mathbb{N}$

$$\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^k\binom{n}{k} + \ldots + 2^n\binom{n}{n} = 3^n.$$

Solution. The binomial theorem states that

$$x^0y^n\binom{n}{0} + xy^{n-1}\binom{n}{1} + x^2y^{n-2}\binom{n}{2} + \ldots + x^ky^{n-k}\binom{n}{k} + \ldots + x^ny^0\binom{n}{n} = (x+y)^n,$$

evaluating at $x = 2$ and $y = 1$ we obtain the above identity. □

2. (20 points) Consider a 5×6 rectangular grid as depicted in Figure 1. A staircase walk is a path in the grid which moves only *right* or *up*.

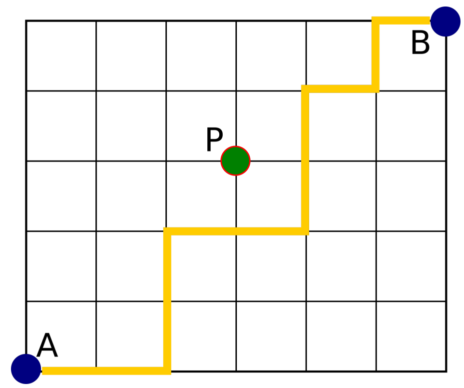

Figure 1: The 5×6 grid and a valid staircase walk from A to B avoiding P.

(a) (10 points) Find the number of staircase walks from A to B.

Solution. There are $\binom{5+6}{6} = \binom{11}{6} = 462$ staircase walks from A to B, since we have a 11 unit walk and we decide in which of the 11 units we take our 6 rights. □

(b) (10 points) How many staircase walks from A to B avoid the point P ?

There are $\binom{6}{3} \cdot \binom{5}{2} = 20 \cdot 10 = 200$ staircase walks that pass through P, since there are $\binom{6}{3}$ paths from A to P, and $\binom{5}{2}$ paths from P to b. Thence, there are

$$\binom{11}{6} - \binom{6}{3} \cdot \binom{5}{2} = 462 - 200 = 262,$$

staircase walks from A to B which avoid P. □

3. (20 points) Solve the following two questions.

(a) (10 points) Compute the number of positive integers $n \in \mathbb{N}$ such that $1 \leq n \leq 10000$, and which are **not** divisible by $6, 7, 9$.

Solution. Let A_6, A_7 and A_9 be the sets of such integers divisible by $6, 7$ and 9. There are $\left\lfloor \frac{10000}{6} \right\rfloor = 1666$ such integers which are divisible by 6, $\left\lfloor \frac{10000}{7} \right\rfloor = 1428$ such numbers which are divisible by 7 and $\left\lfloor \frac{10000}{9} \right\rfloor = 1111$ such integers which are divisible by 9, and thus

$$|A_6| = 1666, \quad |A_7| = 1428, \quad |A_9| = 1111.$$

The desired answer is $10000 - |A_6 \cup A_7 \cup A_9|$. We compute $|A_6 \cup A_7 \cup A_9|$ via the Inclusion-Exclusion Principle. Since we have

$$|A_6 \cap A_7| = \left\lfloor \frac{10000}{42} \right\rfloor = 238, \quad |A_6 \cap A_9| = \left\lfloor \frac{10000}{lcm(6,9)} \right\rfloor = \left\lfloor \frac{10000}{18} \right\rfloor = 555,$$

$$|A_7 \cap A_9| = \left\lfloor \frac{10000}{63} \right\rfloor = 158,$$

and also $\left\lfloor \frac{10000}{lcm(6,7,9)} \right\rfloor = \left\lfloor \frac{10000}{126} \right\rfloor = 79$, we conclude that

$$|A_6 \cup A_7 \cup A_9| = 1666 + 1428 + 1111 - 238 - 555 - 158 + 79 = 3333.$$

Thus, the requested answer is $10000 - 3262 = 6738$. $\square$

(b) (10 points) Prove that in any group of six people there are either three people who all know each other, or three people who are mutual strangers.

Solution. Consider a person A, by the pigeonhole principle that person either *knows* at least three people P_1, P_2 and P_3 or does *not know* at least three people, which we also call P_1, P_2 and P_3.

In the former case, either these three people do not know each other (and so we are done since P_1, P_2 and P_3 satisfies the requirement), or two of them, say P_1 and P_2, know each other, thus yielding three people, P_1, P_2 and A, who know each other.

In the latter case, one proceed analogously. $\square$

4. (20 points) Consider a French deck of 52 cards, containing 4 suits with 13 values

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}$$

in each of the four suits. We are dealt five cards from the deck. These five cards constitute our hand.

(a) (10 points) A hand of five cards is said to be a *trio* if it contains *exactly* three cards with the same value, and the remaining two cards have distinct values amongst them. Show that the probability of having a trio is approximately 0.021128.

Solution. The total number of hands of five cards is $\binom{52}{5} = 2598960$. Let us count how many of such hands can be trios. For that we select a value for the trio, which gives $\binom{13}{1} = 13$, a suit for the trio, which gives a 4. Then we select the remaining two cards, which have two possible distinct values, giving $\binom{12}{2} = 66$, since they cannot be of the same value as the trio, and their possible suits contribute with 4^2. The probability of a trio is thus

$$\frac{13 \cdot 4 \cdot 66 \cdot 4^2}{\binom{52}{5}} = \frac{54912}{2598960} = 0.021128451.$$

(b) (10 points) A hand of five cards is said to be a *full house* if it contains *exactly* three cards with the same value, and the remaining two cards have the same value. Compute the probability of having a full house.

Solution. The same reasoning as above yields a total of $13 \cdot 4 \cdot 12 \cdot \binom{4}{2}$ hands with a full house. This the probability of having a full house is

$$\frac{13 \cdot 4 \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960} = 0.00144057623.$$

5. (20 points) Let us consider a die with six faces, each face with a value in $\{1, 2, 3, 4, 5, 6\}$. Roll the die four consecutive times. Compute the probability that we will see a 3 in at least one of these four rolls.

Solution. The set of all possible outcomes of rolling a die four times has size $6^4 = 1296$, if we consider that order matters. Let X_1 be the subset consisting of those four rolls such that the first roll is a 3. Similarly, let X_2 (and X_3 and X_4) be the subsets consisting of those four rolls such that the second (the third or the fourth) roll is a 3.

We are interested in the cardinality of $|X_1 \cup X_2 \cup X_3 \cup X_4|$, and we can use the Inclusion-Exclusion Principle to compute this cardinality. For that we require the following sizes:

$$|X_1| = |X_2| = |X_3| = |X_4| = 6^3 = 216,$$
$$|X_i \cap X_j| = 6^2 = 36, \quad \forall i, j \in \{1, 2, 3, 4\},$$
$$|X_i \cap X_j \cap X_k| = 6^1 = 6, \quad \forall i, j, k \in \{1, 2, 3, 4\},$$
$$|X_1 \cap X_2 \cap X_3 \cap X_4| = 6^0 = 1.$$

Thus the cardinality is

$$|X_1 \cup X_2 \cup X_3 \cup X_4| = 4 \cdot 216 - \binom{4}{2} \cdot 36 + 4 \cdot 6 - 1 = 671,$$

and the probability is $\frac{671}{1296} = 0.51774691358$. □

Solution II. There is an alternative solution, which does *not* use the Inclusion-Exclusion Principle, in that the union involved consists of disjoint sets. The required probability can be computed as follows. First, the complement of the set X is

$$X^c = (X_1 \cup X_2 \cup X_3 \cup X_4)^c = X_1^c \cap X_2^c \cap X_3^c \cap X_4^c,$$

and since the sets X_i^c, for $i = 1, 2, 3$ and 4, count independent outcomes, each with probability $1 - \frac{1}{6}$. Then we have that the probability of that intersection is

$$\left(1 - \frac{1}{6}\right)^4 = 0.48225308642.$$

Since we want the *opposite* event, the required probability is

$$1 - \left(1 - \frac{1}{6}\right)^4 = 1 - 0.48225308642 = 0.51774691358,$$

yielding the same result as above. □