University of California Davis **Name (Print):** ____________
Euclidean Geometry MAT 141 **Student ID (Print):** ____________

Final Examination **March 17 2020**
Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You *must* put your initials on the top of every scanned page, as well as the Problem number. You are to submit this *take home* Final Examination for MAT 141 Euclidean and Non-Euclidean Geometry via **Gradescope by Tuesday March 17 at 3:10pm**. Please make sure to write clearly and that the quality of the scanning allows for clear readability. You are *required to show your work on each problem* on this exam. You are *not* allowed to discuss the problems of this examination with *anyone* until March 17 at 3:10pm.

You *are* allowed to use an online calculator (such as Wolfram Alpha[1]), your lecture notes from class, the textbook, the Problem Sets, Practice Exams and their Solutions. It is strongly *not recommended* to browse on the Internet, as it will significantly slow you down and the problems are written so that they cannot be googled in two hours. You are *not* allowed to ask *anybody* about these problems, nor ask in any conversations, calls, forums, chats or any medium that involves another human providing hints or answers to you.

You must complete the exam yourself and on your own, and you are strictly forbidden to have *anybody* else answer for you in any capacity. Failure to abide by these rules, including a suspicion of such failure from the graders, will result in a zero grade for this Final Exam.

Regardless of the sources being used, your submitted written answers should be complete and self-contained, never referring to any external resource, nor using mysterious knowledge coming from the Internet. The grade will *solely* be based on the work shown in your submitted answers.

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain clearly why the theorem may be applied. You should cite the name or number of the Lemma, Proposition, or Theorem clearly.

(B) **Organize your work**, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

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[1] https://www.wolframalpha.com/

1. (20 points) (**Euclidean Geometry**) Consider the four Euclidean lines in $\mathbb{R}^2$ given by $L = \{(x, y) \in \mathbb{R}^2 : x = 0\}, M = \{(x, y) \in \mathbb{R}^2 : x = 1\}, N = \{(x, y) \in \mathbb{R}^2 : x = 2\}$ and $R = \{(x, y) \in \mathbb{R}^2 : y = 0\}$.

   (a) (5 points) Find the image of the three points $(1, 1), (1, 0), (0, 0) \in \mathbb{R}^2$ under the isometry $f = \overline{r}_R \circ \overline{r}_N \circ \overline{r}_M \circ \overline{r}_L$.

   (b) (5 points) Show that $f = \overline{r}_R \circ \overline{r}_N \circ \overline{r}_M \circ \overline{r}_L$ is a rotation $R_{P,\theta}$. Find its center $P \in \mathbb{R}^2$ and the angle $\theta \in [0, 2\pi)$.

   (c) (10 points) Let $S = \{(x, y) \in \mathbb{R}^2 : y = -x\} \subseteq \mathbb{R}^2$. Show that the composition $(\overline{r}_S \circ f)$ is a glide reflection, and find $\alpha \in \mathbb{R}^2$ and $K \subseteq \mathbb{R}^2$ a line such that

$$\overline{r}_S \circ f = t_\alpha \circ \overline{r}_K.$$

2. (20 points) (**Spherical Geometry I**) Consider the 2-sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and its points $P = (0,0,1)$, $Q = (0,-1,0)$.

(a) (5 points) Draw the three following sets in the 2-sphere S^2:

$$C_1 = \{R \in S^2 : d_{S^2}(P,R) = 1\}, \quad C_2 = \{R \in S^2 : d_{S^2}(P,R) = 2\},$$

$$C_3 = \{R \in S^2 : d_{S^2}(P,R) = \pi/2\}.$$

Prove that C_1, C_2 are not spherical lines, and prove that C_3 is a spherical line.

(b) (5 points) Find the unique spherical line $L \subseteq S^2$ containing $P, Q \in S^2$.

(c) (10 points) Find the image $(\overline{r}_{C_3} \circ \overline{r}_L)(P)$ of any point $P = (x,y,z) \in S^2$ under the isometry $\overline{r}_{C_3} \circ \overline{r}_L \in \mathrm{Iso}(S^2)$.

3. (20 points) (**Spherical Geometry II**) Consider the unit 2-sphere S^2 and the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ from the North Pole $N = (0,0,1)$. Consider the two points $P_1 = \frac{1}{\sqrt{2}}(1,0,1), P_2 = \frac{1}{\sqrt{3}}(1,-1,1)$, and the spherical line

$$L_{12} = \{T \in S^2 : d_{S^2}(P_1, T) = d_{S^2}(P_2, T)\} \subseteq S^2,$$

of points equidistant to $P_1, P_2 \in S^2$.

(a) (5 points) Compute the distances $d_{S^2}(P_1, P_2)$ and $d_{\mathbb{R}^2}(\pi_N(P_1), \pi_N(P_2))$ and deduce that π_N is not an isometry.

(b) (5 points) Give an example of four points $Q_1, Q_2, R_1, R_2 \in S^2$ such that

$$d_{S^2}(Q_1, Q_2) < d_{\mathbb{R}^2}(\pi_N(Q_1), \pi_N(Q_2)), \text{ and } d_{S^2}(R_1, R_2) > d_{\mathbb{R}^2}(\pi_N(R_1), \pi_N(R_2)).$$

(c) (5 points) Show that $P_1, P_2, N \notin L_{12}$, where $N = (0,0,1) \in S^2$ is the North pole, i.e. prove that none of the points $P_1, P_2, N \in S^2$ belong to L_{12}.

(d) (5 points) Is the image $\pi_N(L_{12}) \subseteq \mathbb{R}^2$ of the spherical line L_{12}, an Euclidean line, an Euclidean circle, or neither ?

4. (20 points) (**Hyperbolic Geometry in $\mathbb{H}^2$**) Let us consider the hyperbolic half-plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} = \{(x, y) \in \mathbb{R}^2 : y > 0\},$$

and the two points $P = -1 + i = (-1, 1)$ and $Q = 2 + 2i = (2, 2)$.

(a) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$.

(b) (5 points) Compute the hyperbolic length $l(\gamma)$ of the Euclidean line segment connecting P, Q, i.e. the region of the line $\{(x, y) \in \mathbb{H}^2 : -x + 3y = 4\} \subseteq \mathbb{H}^2$ bounded by the two points $P, Q \in \mathbb{H}^2$. Is it true that $l(\gamma) = d_{\mathbb{H}^2}(P, Q)$?

(c) (10 points) Find the unique hyperbolic line $L \subseteq \mathbb{H}^2$ containing $P, Q \in \mathbb{H}^2$.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

   (a) (2 points) Let $L \subseteq \mathbb{H}^2$ be a hyperbolic line and $P \in \mathbb{H}^2$. There exists a unique line $M \subseteq \mathbb{H}^2$ parallel to L and passing through P.

   (1) True. (2) False.

   (b) (2 points) Every isometry $f \in \mathrm{Iso}(S^2)$ must have *at least* two fixed points $P, Q \in S^2$, i.e. there exists $P, Q \in S^2$ distinct with $f(P) = P$ and $f(Q) = Q$.

   (1) True. (2) False.

   (c) (2 points) There exists an isometry $g \in \mathrm{Iso}(\mathbb{R}^2)$ which does *not* preserve angles.

   (1) True. (2) False.

   (d) (2 points) Let $P, Q \in S^2$ be two points in the 2-sphere with distance $d_{S^2}(P, Q) < \pi$. There exists a unique line $L \subseteq S^2$ containing P and Q.

   (1) True. (2) False.

   (e) (2 points) Let $h \in \mathrm{Iso}(\mathbb{R}^2)$ be an isometry which fixes every point in the line $L = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $h((1, 0)) = (0, 1)$. Then it must be that $h = \overline{r}_L$.

   (1) True. (2) False.

   (f) (2 points) The curve $L = \{(x, y) \in \mathbb{H}^2 : y = 1\} \subseteq \mathbb{H}^2$ is a hyperbolic line.

   (1) True. (2) False.

   (g) (2 points) The hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$ is always less than or equal to the Euclidean distance $d_{\mathbb{R}^2}(P, Q)$ for any two points $P, Q \in \mathbb{H}^2$.

   (1) True. (2) False.

   (h) (2 points) For any glide reflection $t_\alpha \circ \overline{r}_L \in \mathrm{Iso}(\mathbb{R}^2)$, we must have that the composition $t_\alpha \circ \overline{r}_L = \overline{r}_L \circ t_\alpha$ commutes.

   (1) True. (2) False.

   (i) (2 points) Let $R_{P,\theta}, R_{Q,\psi} \in \mathrm{Iso}(\mathbb{R}^2)$ be two arbitrary rotations. Then we must have that $R_{P,\theta} \circ R_{Q,\psi} = R_{Q,\psi} \circ R_{P,\theta}$.

   (1) True. (2) False.

   (j) (2 points) There exists two curves $\gamma_1, \gamma_2 \subseteq \mathbb{H}^2$ such that the hyperbolic length $l_{\mathbb{H}^2}(\gamma_1)$ of γ_1 is greater than the Euclidean length $l_{\mathbb{R}^2}(\gamma_1)$ of γ_1, and the hyperbolic length $l_{\mathbb{H}^2}(\gamma_2)$ of γ_2 is less than the Euclidean length $l_{\mathbb{R}^2}(\gamma_2)$ of γ_2.

   (1) True. (2) False.