University of California Davis **Name (Print):** ____________________
Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Solutions to Midterm Examination **February 7 2020**
Time Limit: 50 Minutes

This examination document contains 9 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the points $O = (0,0), P = (1,0), Q = (1,1) \in \mathbb{R}^2$ in the Euclidean plane. Let $L, M \subseteq \mathbb{R}^2$ be the lines such that $O, P \in L$ and $O, Q \in M$.

   (a) (5 points) Find the image of the point of the point $(-4.5, 9) \in \mathbb{R}^2$ under the isometry $\overline{r}_M \circ \overline{r}_L$ given by the reflections along L, M.

   The intersection point of L, M is the origin $O = L \cap M$, and the angle between L and M is $\pi/4$. Thus $R_{O,\pi/2} = \overline{r}_M \circ \overline{r}_L$ and the image of $(-4.5, 9) \in \mathbb{R}^2$ under the isometry is $(-9, -4.5) \in \mathbb{R}^2$.

   Alternatively, the reflection $\overline{r}_L$ acts as $\overline{r}_L(x, y) = (x, -y)$ and the reflection $\overline{r}_M$ acts as $\overline{r}_L(x, y) = (y, x)$. In consequence,

$$(\overline{r}_M \circ \overline{r}_L)(x, y) = \overline{r}_M(x, -y) = (-y, x),$$

   and $(\overline{r}_M \circ \overline{r}_L)(-4.5, 9) = (-9, -4.5)$.

   (b) (5 points) Show that the isometry given by the composition

$$(\overline{r}_M \circ \overline{r}_L)^4 = \overline{r}_M \circ \overline{r}_L \circ \overline{r}_M \circ \overline{r}_L \circ \overline{r}_M \circ \overline{r}_L \circ \overline{r}_M \circ \overline{r}_L$$

   is the identity.

   Since $R_{O,\pi/2} = \overline{r}_M \circ \overline{r}_L$, we have

$$(\overline{r}_M \circ \overline{r}_L)^4 = R^4_{O,\pi/2} = R_{O,4\cdot\pi/2} = R_{O,2\pi}.$$

   Given that $R_{O,2\pi} = id$, this implies the statement.
   An alternative solution is to verify that the points O, P, Q are fixed by $(\overline{r}_M \circ \overline{r}_L)^4$, and use that an isometry is uniquely determined by the images of three non-collinear points.

(c) (5 points) Let $N \subseteq \mathbb{R}^2$ be the unique line containing $P, Q \in \mathbb{R}^2$. Show that the isometry $\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L$ is a rotation.

Since $\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L$ is a composition of four reflections, by the Classification Theorem of Euclidean Isometries in $\mathbb{R}^2$, we know that it must be a translation or a rotation. By contradiction, assume this composition is a translation. Given that

$$(\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L)(1, 0) = (0, 1),$$

if it were a translation it should be $t_{(\alpha,\beta)}$ with $(\alpha, \beta) = (0, 1) - (1, 0) = (-1, 1)$. Nevertheless,

$$(\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L)(0, 0) = (0, 2),$$

and thus $\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L$ cannot be a translation, since it should also be $t_{(\alpha,\beta)}$ with $(\alpha, \beta) = (0, 2) - (0, 0) = (0, 2)$.

(d) (5 points) Find a point $C \in \mathbb{R}^2$ and an angle $\theta \in \mathbb{R}$ such that $R_{C,\theta} = \overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L$.

Note that the point $N = (1, 1)$ is fixed, i.e.

$$(\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L)(1, 1) = (1, 1)$$

since $(1, 1) \in M \cap N$. We know that the isometry is a non-trivial rotation by Part (c), and thus it suffices to determine the angle θ. By observing that

$$(\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L)(1, 0) = (0, 1)$$

we conclude

$$\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L = R_{(1,1),-\pi/2}.$$

Alternatively, one can rewrite $\overline{r}_M \circ \overline{r}_L \circ \overline{r}_N \circ \overline{r}_L$ in terms of reflections along different lines until one simplifies to two reflections.

2. (20 points) Let $P = (0,0), Q = (1,0), R = (0,3) \in \mathbb{R}^2$, and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = (0,2)$, $f(Q) = (-1,2)$ and $f(R) = (0,5)$. Consider the points $S = (-3,0)$.

   (a) (5 points) Find the distances $d(P,S), d(Q,S), d(R,S)$.

   The Euclidean distance is $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. Hence

   $$d(P,S) = 3, d(Q,S) = 4, d(R,S) = 3\sqrt{2}.$$

   (b) (5 points) Find the image $f(S)$ of the point $S \in \mathbb{R}^2$.

   We need to find the unique point $f(S)$ such that

   $$d(P,S) = d(f(P),f(S)), d(Q,S) = d(f(Q),f(S)), d(R,S) = d(f(R),f(S)).$$

   Since the point $(3,2) \in \mathbb{R}^2$ satisfies

   $$d(f(P),f(S)) = 3, d(f(Q),f(S)) = 4, d(f(R),f(S)) = 3\sqrt{2},$$

   it must be that $f(S) = (3,2)$.

   Alternatively, one can notice that the glide reflection $\overline{r}_L \circ t_{(0,2)}$, with $L = \{(x,y) : x = 0\} \subseteq \mathbb{R}^2$ the y-axis, sends P, Q, R exactly as the isometry f does. By the characterization of isometries in terms of three non-collinear points, it must be that

   $$f = \overline{r}_L \circ t_{(0,2)},$$

   and from here we conclude $f(S) = (3,2)$.

(c) (5 points) Show that f cannot be a translation $t_{(\alpha,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

The isometry f is orientation reversing, and thus it cannot be a translation (nor a rotation). Alternatively, if f were a translation $f = t_{(\alpha,\beta)}$ we would have

$$t_{(\alpha,\beta)}(P) = t_{(\alpha,\beta)}(0,0) = (\alpha,\beta) = f(P) = (0,2),$$

and thus $(\alpha,\beta) = (0,2)$. Nevertheless, $t_{(\alpha,\beta)}(Q) = (1,2)$, which is not equal to the image $f(Q) = (-1,2)$. Thus f cannot be a translation.

(d) (5 points) Suppose that f has no fixed points, show that f is a glide reflection.

By the Classification Theorem of Euclidean Isometries and Part (c), we conclude that the isometry f must be a reflection or a glide reflection. The hypothesis that f has no fixed points implies that f must be a glide reflection.

3. (20 points) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

(a) (5 points) Draw the Γ-orbits of the two points $P = (1, -2), Q = (-0.1, 0.9) \in \mathbb{R}^2$.

The Γ-orbits consist of the square-grids described as follows:

$$\Gamma P = \{(n, m) \in \mathbb{Z}^2\} \subseteq \mathbb{R}^2, \qquad \Gamma Q = \{(n - 0.1, m + 0.9) \in \mathbb{Z}^2\} \subseteq \mathbb{R}^2.$$

(b) (5 points) Find the distance $d(\Gamma P, \Gamma Q)$ of $\Gamma P, \Gamma Q \in T^2$ in the 2-torus.

Since $(1, -2) = (0, 0) \in T^2$ and $Q = (-0.1, 0.9) = (-0.1, -0.1) \in T^2$ as points in the 2-torus, we obtain $d(P, Q) = \sqrt{0.02}$.

(c) (5 points) Give three different lines $L_0, L_1, L_2 \subseteq T^2$ in the 2-torus such that both points $\Gamma P, \Gamma Q \in T^2$ belong to each L_0, L_1 and L_2.

We can choose the image of the line $L_0 = \{(x, y) \in \mathbb{R}^2 : x = y\} \subseteq \mathbb{R}^2$, which contains $(0, 0), (-0.1, -0.1) \in \mathbb{R}^2$, the image of the line $L_1 = \{(x, y) \in \mathbb{R}^2 : 1.9x = 0.9y\} \subseteq \mathbb{R}^2$, which contains $(0, 0), (0.9, 1.9) \in \mathbb{R}^2$, and the image of the line $L_2 = \{(x, y) \in \mathbb{R}^2 : 2.9x = 0.9y\} \subseteq \mathbb{R}^2$, which contains $(0, 0), (0.9, 2.9) \in \mathbb{R}^2$.

(d) (5 points) Find the number of intersection points between the two lines

$$\{(x, y) \in T^2 : x = 12y\} \subseteq T^2, \quad \{(x, y) \in T^2 : x = 0.3\} \subseteq T^2.$$

There are 12 intersection points, given by the image of the intersections in $\mathbb{R}^2$ of the lines $\{(x, y) : x = 0.3\} \subseteq \mathbb{R}^2$ and $M_i = \{(x, y) : x = 12y - i\} \subseteq \mathbb{R}^2$, for $0 \leq i \leq 11$, $i \in \mathbb{M}$, under the Γ-orbit map.

4. (20 points) Let $M = \mathbb{R}^2/\Gamma$ be the Euclidean Twisted Cylinder, where

$$\Gamma = \langle \overline{r} \circ t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2).$$

is the subgroup generated by the glide reflection $\overline{r} \circ t_{(1,0)}$. Consider the two points $P = (0.2, 0.8), Q = (0.7, -0.8) \in M$ in the twisted cylinder.

(a) (5 points) Compute the distace $d(P, Q)$ between $P, Q \in M$.

Since the Γ-orbit of $Q = (0.7, -0.8)$ contains $(-0.3, 0.8)$, the minimum distance between the Γ-orbits is achieved by the distance $d((0.2, 0.8), (-0.3, 0.8)) = 0.5$.

(b) (5 points) Find an isometry $g : M \longrightarrow M$ such that $g(P) = Q$.

The horizontal translation $t_{(0.5,0)}$ brings Q to the point

$$t_{(0.5,0)}(Q) = (1.2, -0.8) = (0.2, 0.8) = P,$$

as required.

(c) (5 points) Show that the subgroup $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is fixed point free.

The generator $\overline{r} \circ t_{(1,0)}$ of Γ is a glide reflection, which does not have fix points. The composition of a glide reflection with itself is either a translation or a glide reflection. Thus Γ contains no elements with fixed points.

(d) (5 points) Give an element $g \in \Gamma$ which is *not* a glide reflection.

The generator $\overline{r} \circ t_{(1,0)}$ of Γ is equal to $t_{(1,0)} \circ \overline{r}$ since a translation commutes with a reflection along a line in the direction of the translation. Thus, the composition of the generating isometry $\overline{r} \circ t_{(1,0)}$ with itself is

$$t_{(1,0)} \circ \overline{r} \circ \overline{r} \circ t_{(1,0)} = t_{(1,0)} \circ t_{(1,0)} = t_{(2,0)},$$

since $\overline{r}^2 = id$ a reflection squares to the identity. In conclusion $t_{(2,0)} \in \Gamma$ is a translation, and thus it is an element in Γ which is not a glide reflection.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

   (a) (2 points) The composition of two rotations is a rotation.

   (1) True. (2) **False**.

   (b) (2 points) There are no lines $L, N \subseteq M$ in the twisted cylinder with $|L \cap N| = 2$.

   (1) True. (2) **False**.

   (c) (2 points) An isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is different from the identity cannot have more than three fixed points.

   (1) True. (2) **False**.

   (d) (2 points) The set of points equidistant to two distinct points $P, Q \in C$ in the cylinder consists of a line.

   (1) True. (2) **False**.

   (e) (2 points) Let $\Gamma \subseteq \mathbb{R}^2$ be an arbitrary subgroup, then there always exist finitely many fundamental domains $D_\Gamma \subseteq \mathbb{R}^2$.

   (1) True. (2) **False**.

   (f) (2 points) Any composition of an even number of reflections, including zero, can be expressed as a composition of two reflections.

   (1) **True**. (2) False.

   (g) (2 points) For any pair of points $P, Q \in C$ in the cylinder, there are infinitely many distinct lines $L \subseteq C$ containing $P, Q \in C$.

   (1) True. (2) **False**.

   (h) (2 points) Given two points $P, Q \in T^2$ in the 2-torus, there exists an isometry $f : T^2 \longrightarrow T^2$ such that $f(P) = Q$.

   (1) **True**. (2) False.

   (i) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be discontinuous and fixed point free. Then Γ must contain a non-trivial translation.

   (1) **True**. (2) False.

   (j) (2 points) A fixed point free isometry $f \in \text{Iso}(\mathbb{R}^2)$ must be a translation.

   (1) True. (2) **False**.