MAT 141: PROBLEM SET 1

DUE TO FRIDAY JAN 17 AT 10:00AM

Abstract. This is the first problem set for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. It was posted online on Friday Jan 10 and is due Friday Jan 17 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice problems on the geometry of the Euclidean Plane $\mathbb{R}^2$. In particular, we would like to become familiar with the distance function, isometries and composition of isometries.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Consider the Euclidean distance in $\mathbb{R}^2$, i.e. the distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(i) Prove that this distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following three properties:

(a) For any pairs of points $P, Q \in \mathbb{R}^2$,

$$d(P, Q) \geq 0,$$

and equality only occurs if $P = Q$.

(b) For any pairs of points $P, Q \in \mathbb{R}^2$,

$$d(P, Q) = d(Q, P).$$

(c) For any three points $P, Q, R \in \mathbb{R}^2$,
$$d(P,Q) \leq d(Q,R) + d(R,P).$$

(ii) Describe for which triples of points $P, Q, R \in \mathbb{R}^2$ the general inequality
$$d(P,Q) \leq d(Q,R) + d(R,P).$$
that you have proven in Part (i).c is actually an equality.

Problem 2. For each of the following maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, decide whether they are isometries of the Euclidean plane $\mathbb{R}^2$ or not. If they are *not* isometries, provide a counter-example, and if they are, provide a proof.

(a) $f(x,y) = (-2x, x+y)$,

(b) $f(x,y) = (\cos(x), y)$,

(c) $f(x,y) = (x^2, y)$,

(d) $f(x,y) = (y, x)$,

(e) $f(x,y) = (-x, -y)$,

(f) $f(x,y) = (x, xy)$,

Problem 3. (20 pts) Let $P = (3,4) \in \mathbb{R}^2$ be a point and $L \subseteq \mathbb{R}^2$ be the line
$$L = \{(x,y) : y = \sqrt{3}x - \sqrt{3} + 2\}.$$

(a) Let $R_{\pi/3,P}$ be the counter-clockwise rotation by $\pi/3$-radians centered at P. Find a formula for the isometry $R_{\pi/3,P}$.

(b) Where does the point $(-2,-7)$ map under $R_{\pi/3,P}$?

(c) Let $\overline{r}_L$ be the reflection along the line L. Find a formula for the isometry $\overline{r}_L$.

(d) Describe where the points $(1,2)$, $(-2,-7)$ and $(3,4)$ map under the isometry $\overline{r}_L$.

(e) Consider the composition $R_{\pi/3,P} \circ \overline{r}_L$. Where does the origin $(0,0) \in \mathbb{R}^2$ map to ?

(f) Consider the composition $\overline{r}_L \circ R_{\pi/3,P}$. Compute the imagine of the origin $(0,0)$ under this isometry and compare with Part (e).

Problem 4. (20 pts) In this problem we explore basic compositions of rotations and translations. Solve the following parts:

(a) Let $\theta, \phi \in S^1$ be two angles. Show that

$$R_\theta \circ R_\phi = R_{\theta+\phi}.$$

(b) Let $\theta \in S^1$ be an angle. Find the unique angle $\phi \in S^1$ such that $R_\theta \circ R_\phi = \mathrm{Id}$ is the identity map $\mathrm{Id}(x, y) = (x, y)$.

(c) Let (α, β) and (γ, δ) be two points in the Euclidean Plane $\mathbb{R}^2$. Prove that

$$t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)}.$$

(d) Let $(\alpha, \beta) \in \mathbb{R}^2$ be a point in Euclidean plane. Find the unique $(\gamma, \delta) \in \mathbb{R}^2$ such that the composition $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = \mathrm{Id}$.

Problem 5. (20 pts) Let $L = \{(x, y) : y = 0\} \subseteq \mathbb{R}^2$ and $M = \{(x, y) : x = 0\} \subseteq \mathbb{R}^2$ be the x and y-axis respectively.

(a) Show that $\overline{r}_L \overline{r}_M(x, y) = (-x, -y)$.

(b) Prove that there exists no line $N \subseteq \mathbb{R}^2$ such that

$$\overline{r}_N = \overline{r}_L \overline{r}_M,$$

where L, M are as in Part (a). Thus, we learn that the composition of reflections is *not* always a reflection.

(c) Find an angle $\phi \in S^1$ such that the composition $\overline{r}_L \overline{r}_M$ in Part (a) equals the rotation R_ϕ, i.e. $R_\phi = \overline{r}_L \overline{r}_M$. Thus, we learn that the composition of reflections can *sometimes* be a rotation.

(d) Find all the angles $\theta \in S^1$, if any, such that the rotation R_θ, centered at the origin, *commutes* with any reflection $\overline{r}_L$, where L is a line through the origin:

$$R_\theta \circ \overline{r}_L = \overline{r}_L \circ R_\theta.$$

Problem 6. (20 pts) Consider the square $S \subseteq \mathbb{R}^2$ with four vertices given by the points $(-1, -1), (-1, 1), (1, 1), (1, -1) \in \mathbb{R}^2$. The square consists of the convex hull of these four points, i.e. the region given by

$$S = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

(a) Show that there exists *no* translation $t_{(\alpha,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, except for the identity, such that $t_{(\alpha,\beta)}(S) \subseteq S$, i.e. the translation sends the square to the square.

(b) Find *four* distinct lines $L_1, L_2, L_3, L_4 \subseteq \mathbb{R}^2$ such that the reflections $\overline{r}_{L_i}$, $1 \leq i \leq 4$, all satisfy the inclusion $\overline{r}_{L_i}(S) \subseteq S$.

(c) Find 27 distinct lines $M \subseteq \mathbb{R}^2$ such that $\overline{r}_M(S) \not\subseteq S$, i.e. the reflection $\overline{r}_M$ maps the square S *not inside* the square S.

(d) Find all angles $\theta \in S^1$ such that $R_\theta(S) \subseteq S$.

(e) Find *infinitely many* angles $\theta \in S^1$ such that $R_\theta(S) \not\subseteq S$.

Problem 7. (20 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) The linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is an isometry.

(b) Any linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with $a \neq 0$, must be an isometry.

(c) The composition $f \circ g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of two isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is always an isometry.

(d) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry. If there exist infinitely many points $P \in \mathbb{R}^2$ such $f(P) = P$, then $f = \mathrm{Id}$ must be the identity.

(e) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isometry which fixes the points $(0, 0)$, $(1, 0)$ and $(0, 1)$, i.e. $f(0, 0) = (0, 0)$, $f(1, 0) = (1, 0)$ and $f(0, 1) = (0, 1)$. Then $f = \mathrm{Id}$ must be the identity.

(f) The composition of reflections is *always* a reflection.

(g) The composition of rotations centered at the origin are *always* rotations.

(h) The composition of translations is *always* a translation.

(i) There is an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that sends the square S, as defined in Problem 6, strictly inside itself, i.e.

$$f(S) \subseteq \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\}.$$

(j) For any rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, there exists a power $n \in \mathbb{N}$ such that the composition $R_\theta^n = \mathrm{Id}$.