SOLUTIONS TO PROBLEM SET 1

MAT 141

Abstract. These are the solutions to Problem Set 1 for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. The problems were posted online on Friday Jan 10 and due Friday Jan 17 at 10:00am.

Problem 1. Consider the Euclidean distance in $\mathbb{R}^2$, i.e. the distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(i) Prove that this distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following three properties:

(a) For any pairs of points $P, Q \in \mathbb{R}^2$,

$$d(P, Q) \geq 0,$$

and equality only occurs if $P = Q$.

(b) For any pairs of points $P, Q \in \mathbb{R}^2$,

$$d(P, Q) = d(Q, P).$$

(c) For any three points $P, Q, R \in \mathbb{R}^2$,

$$d(P, Q) \leq d(Q, R) + d(R, P).$$

(ii) Describe for which triples of points $P, Q, R \in \mathbb{R}^2$ the general inequality

$$d(P, Q) \leq d(Q, R) + d(R, P).$$

that you have proven in Part (i).c is actually an equality.

Solution.

(i) These properties (positive, symmetric, and triangle inequality) are the most important qualities of any distance function. In a general metric space, any function that behaves like this is a valid notion of "distance".

(a) In coordinates, write $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. The distance function is always nonnegative, because it is the (positive) square root of the sum of two nonnegative terms $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$. Suppose we have equality:

$d(P,Q)=0$. Then

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=0.$$

Squaring both sides, we have

$$(x_2-x_1)^2+(y_2-y_1)^2=0.$$

The only way that two nonnegative terms can sum to zero is if both of them are identically zero, so

$$(x_2-x_1)^2=0 \qquad \text{and} \qquad (y_2-y_1)^2=0.$$

Rearranging, we see that $x_2=x_1$ and $y_2=y_1$, so $P=Q$.

(b) For any real number $t\in\mathbb{R}$, we have $(-t)^2=t^2$, so

$$(x_1-x_2)^2=(-(x_2-x_1))^2=(x_2-x_1)^2,$$

and similarly, $(y_1-y_2)^2=(y_2-y_1)^2$. Using these, we calculate

$$\begin{aligned} d(P,Q) &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= d(Q,P). \end{aligned}$$

(c) We are free to perform any isometries we want before doing our analysis (replacing P, Q, and R with their images), because any isometry will not change the distances that we care about. Therefore, by performing a translation so that P goes to the origin, and then performing a rotation so that (the new image of) R lands on the positive x-axis, we may assume that $P=(0,0)$ and $R=(x_3,0)$. If you like, we've taken three points in ambient space and just chosen a convenient coordinate system in which to work. Our calculations are now easier.

Replacing $d(P,Q)$, $d(Q,R)$, and $d(R,P)$ with their formulas, we see that we need to prove

$$\sqrt{x_2^2+y_2^2} \stackrel{?}{\leq} \sqrt{(x_3-x_2)^2+y_2^2}+x_3,$$

where the "?" is to remind you that this isn't derived yet, but something that we want to show. The proof is complete when we can manipulate this formula into something that is obviously true. First, square both sides, expand, and combine terms, giving

$$\begin{aligned} x_2^2+y_2^2 &\stackrel{?}{\leq} ((x_3-x_2)^2+y_2^2)+2x_3\sqrt{(x_3-x_2)^2+y_2^2}+x_3^2 \\ &= 2x_3^2+x_2^2-2x_3x_2+y_2^2+2x_3\sqrt{(x_3-x_2)^2+y_2^2}. \end{aligned}$$

Cancelling terms, rearranging, and dividing by 2, we now have

$$x_3x_2-x_3^2 \stackrel{?}{\leq} x_3\sqrt{(x_3-x_2)^2+y_2^2}.$$

If $x_3 = 0$, then our original inequality is already true (make sure you see why), so now assume $x_3 \neq 0$. Divide both sides by x_3, giving

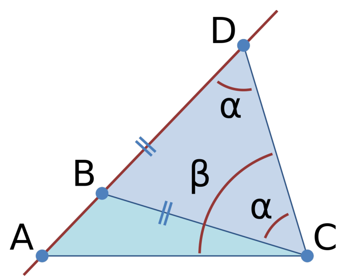
$$x_2 - x_3 \overset{?}{\leq} \sqrt{(x_3 - x_2)^2 + y_2^2}.$$

Now square both sides again. We finally have

$$(x_2 - x_3)^2 \leq (x_3 - x_2)^2 + y_2^2,$$

which is obviously true, so the original inequality was true.

Alternatively, recall Euclid's proof from Discussion 1:



We want to show $AC < AB+BC$. Place a fourth point D on the line containing AB, such that $BD = BC$. Then $\triangle DBC$ is an isosceles triangle, so $\angle BDC = \angle DCB$, which we call α. Notice then that the angle β is larger than α. Compare β to the copy of α at D. Since $\beta > \alpha$, the sides opposite to these angles are also in the same relation: $AD > AC$. Then

$$AB + BC = AB + BD = AD > AC,$$

which is what we wanted.

(ii) Notice that in Euclid's proof above, the inequality is strict ($<$ instead of $\leq$). That's because you never get equality in an actual triangle. Equality is achieved if and only if P, Q, and R lie on a common line with R either between P and Q or the same as P or Q.

Here's a proof. Suppose all our points are distinct. If they are not collinear, then Euclid's construction above shows that we don't have equality in the triangle inequality. If they are collinear but R is not in the middle, then either $d(Q, R)$ (if P is in the middle) or $d(R, P)$ (if Q is in the middle) is strictly the largest of our three distances, so we don't have equality. If they are collinear with R in the middle, then equality follows from a quick picture.

Next, suppose two points coincide, with the third distinct. If P and Q are the coincident points, then the left side of the inequality is zero, but the right side isn't, so we don't have equality. If R coincides with P, then our formula reads $d(P, Q) \leq d(Q, R) + 0 = d(Q, P)$, which forces equality, and similarly if R coincides with Q. Finally, if all three points are the same, then all distances are zero, and we have equality.

Problem 2. For each of the following maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, decide whether they are isometries of the Euclidean plane $\mathbb{R}^2$ or not. If they are *not* isometries, provide a counter-example, and if they are, provide a proof.

(a) $f(x, y) = (-2x, x + y)$,

(b) $f(x, y) = (\cos(x), y)$,

(c) $f(x, y) = (x^2, y)$,

(d) $f(x, y) = (y, x)$,

(e) $f(x, y) = (-x, -y)$,

(f) $f(x, y) = (x, xy)$,

Solution. Recall the definition: A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called an **isometry** if, for any points $P, Q \in \mathbb{R}^2$, we have

$$d(f(P), f(Q)) = d(P, Q).$$

To prove f is an isometry, we prove it for general points P and Q, and to prove f is not an isometry, we produce a specific pair of points for which the equation above fails.

(a) This is not an isometry because $d((1, 0), (0, 0)) = 1$, while

$$d(f(1, 0), f(0, 0)) = d((-2, 1), (0, 0)) = \sqrt{2^2 + 1^2} = \sqrt{5} \neq 1.$$

(b) This is not an isometry because $d((\pi, 0), (0, 0)) = \pi$, while

$$d(f(\pi, 0), f(0, 0)) = d((-1, 0), (1, 0)) = 2 \neq \pi.$$

(c) This is not an isometry because $d((2, 0), (0, 0)) = 2$, while

$$d(f(2, 0), f(0, 0)) = d((4, 0), (0, 0)) = 4 \neq 2$$

(d) This is reflection through the line $y = x$. It is an isometry because, for two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ in the plane, we have

$$\begin{aligned} d(f(x_1, y_1), f(x_2, y_2)) &= d((y_1, x_1), (y_2, x_2)) \\ &= \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= d((x_1, y_1), (x_2, y_2)). \end{aligned}$$

(e) This is a rotation through π about the origin. It is an isometry because, for two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ in the plane, we have

$$\begin{aligned} d(f(x_1, y_1), f(x_2, y_2)) &= d((-x_1, -y_1), (-x_2, -y_2)) \\ &= \sqrt{(-x_2 + x_1)^2 + (-y_2 + y_1)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= d((x_1, y_1), (x_2, y_2)). \end{aligned}$$

(f) This is not an isometry because $d((0, 1), (0, 0)) = 1$, while

$$d(f(0, 1), (0, 0)) = d((0, 0), (0, 0)) = 0 \neq 1.$$

Problem 3. (20 pts) Let $P = (3, 4) \in \mathbb{R}^2$ be a point and $L \subseteq \mathbb{R}^2$ be the line

$$L = \{(x, y) : y = \sqrt{3}x - \sqrt{3} + 2\}.$$

(a) Let $R_{\pi/3,P}$ be the counter-clockwise rotation by $\pi/3$-radians centered at P. Find a formula for the isometry $R_{\pi/3,P}$.

(b) Where does the point $(-2, -7)$ map under $R_{\pi/3,P}$?

(c) Let $\overline{r}_L$ be the reflection along the line L. Find a formula for the isometry $\overline{r}_L$.

(d) Describe where the points $(1, 2)$, $(-2, -7)$ and $(3, 4)$ map under the isometry $\overline{r}_L$.

(e) Consider the composition $R_{\pi/3,P} \circ \overline{r}_L$. Where does the origin $(0, 0) \in \mathbb{R}^2$ map to ?

(f) Consider the composition $\overline{r}_L \circ R_{\pi/3,P}$. Compute the imagine of the origin $(0, 0)$ under this isometry and compare with Part (e).

Solution.

(a) To rotate about a point P different from the origin, we translate that point to the origin, rotate about the origin in the usual way, and then translate back to P. In formulas,

$$R_{\theta,P} = t_P R_\theta t_P^{-1} = t_P R_\theta t_{-P},$$

where R_θ is the usual linear isometry of counter-clockwise rotation by θ about the origin, and t_P is the isometry of translation by P. This is called the *conjugate* of R_θ by t_P. Think of it like you would the change-of-basis formula from linear algebra.

Therefore,

$$R_{\pi/3,(3,4)} = t_{(3,4)} R_{\pi/3} t_{(-3,-4)}.$$

Let's apply these one by one to a point (x, y) (remember that function composition is read right to left!). First, the translation $t_{(-3,-4)}$ produces new points

$$x' = x - 3$$
$$y' = y - 4.$$

Next, remember that the functionR_θ is the same as multiplying a column vector on the left by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which, for our case of $\theta = \pi/3$ becomes

$$\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix},$$

so applying this function to our new coordinates gives

$$x'' = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' = \frac{1}{2}(x-3) - \frac{\sqrt{3}}{2}(y-4)$$
$$y'' = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \frac{\sqrt{3}}{2}(x-3) + \frac{1}{2}(y-4).$$

Finally, we apply $t_{(3,4)}$

$$x''' = x'' + 3 = \frac{1}{2}(x-3) - \frac{\sqrt{3}}{2}(y-4) + 3$$
$$y''' = y'' + 4 = \frac{\sqrt{3}}{2}(x-3) + \frac{1}{2}(y-4) + 4,$$

so our formula is

$$\boxed{R_{\pi/3,P}(x,y) = \left(\frac{1}{2}(x-3) - \frac{\sqrt{3}}{2}(y-4) + 3, \frac{\sqrt{3}}{2}(x-3) + \frac{1}{2}(y-4) + 4\right)}.$$

Notice in particular that the point $P = (3, 4)$ is fixed by this function, as it should be.

(b) Plugging in $(x, y) = (-2, -7)$ above, we have

$$R_{\pi/3,P}(-2,-7) = \left(\frac{1 + 11\sqrt{3}}{2}, \frac{-3 - 5\sqrt{3}}{2}\right).$$

(c) The formula for $\overline{r}_L$ can again be found by conjugation, but here is a more straightforward approach: to reflect a point (x_0, y_0) about a line L given by $y = mx + b$, first draw the perpendicular line to L passing through (x_0, y_0), which has equation

$$y = -\frac{1}{m}(x - x_0) + y_0.$$

This crosses line L at the x-value

$$\hat{x} = \frac{m(y_0 - b) + x_0}{m^2 + 1} = x_0 + \frac{m}{m^2 + 1}(y_0 - b - mx_0).$$

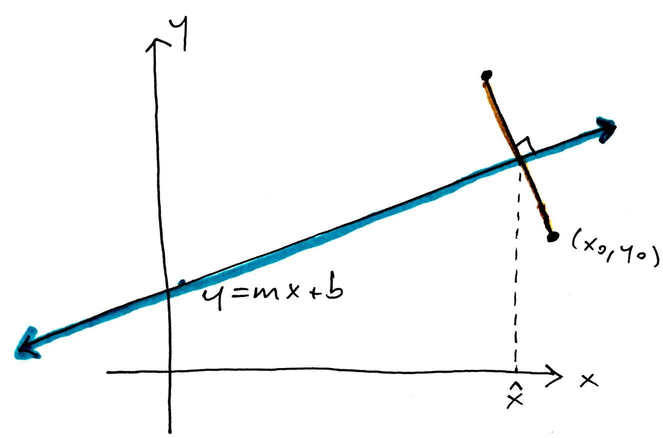
The reflection point is twice as far (along our perpendicular) from (x_0, y_0) as is this intersection, so the x-value of the reflection of (x_0, y_0) about L is

$$x_0 + \frac{2m}{m^2 + 1}(y_0 - b - mx_0).$$

Plugging this in for x in our equation for the perpendicular, we get the corresponding y-value, showing that reflection through L is given by

$$(x, y) \mapsto \left(x + \frac{2m}{m^2 + 1}(y - b - mx), y - \frac{2}{m^2 + 1}(y - b - mx)\right), \tag{1}$$

where we have removed the subscripts to call (x_0, y_0) simply (x, y). (Note, this construction cannot apply to vertical lines.)



For our purposes, we have $m = \sqrt{3}$ and $b = -\sqrt{3} + 2$, so

$$\boxed{\overline{r}_L(x, y) = \left(\frac{-x + \sqrt{3}y - 2\sqrt{3} + 3}{2}, \frac{\sqrt{3}x + y - \sqrt{3} + 2}{2}\right)}.$$

For a quick "sanity check", notice that

$$\overline{r}_L(x, \sqrt{3}x - \sqrt{3} + 2) = (x, \sqrt{3}x - \sqrt{3} + 2),$$

so L is fixed by $\overline{r}_L$, as you'd expected.

(d) The point $(1, 2)$ lies on L, so it is fixed by $\overline{r}_L$. For the others, we calculate

$$\overline{r}_L(-2, -7) = \left(\frac{5 - 9\sqrt{3}}{2}, \frac{-5 - 3\sqrt{3}}{2}\right) \qquad \text{and} \qquad \overline{r}_L(3, 4) = (\sqrt{3}, 3 + \sqrt{3}).$$

(e) We calculate using the formulas from parts (a) and (c):

$$\begin{aligned} R_{\pi/3,P} \circ \overline{r}_L(0,0) &= R_{\pi/3,P}(\overline{r}_L(0,0)) \\ &= R_{\pi/3,P}\left(\frac{3}{2} - \sqrt{3}, 1 - \frac{\sqrt{3}}{2}\right) \\ &= \left(\frac{21+2\sqrt{3}}{4}, \frac{10-7\sqrt{3}}{4}\right). \end{aligned}$$

(f) We calculate using the formulas from parts (a) and (c):

$$\begin{aligned} \overline{r}_L \circ R_{\pi/3,P}(0,0) &= \overline{r}_L(R_{\pi/3,P}(0,0)) \\ &= \overline{r}_L\left(\frac{3}{2} + 2\sqrt{3}, 2 - \frac{3\sqrt{3}}{2}\right) \\ &= \left(-\frac{3}{2} - \sqrt{3}, 5 - \frac{\sqrt{3}}{2}\right). \end{aligned}$$

Notice that this differs from the image of the origin in part (e). That is, reflection in L does not *commute* with rotation about P.

Remark: In the language of abstract algebra, we say that the group of isometries of the plane, $\mathrm{Iso}(\mathbb{R}^2)$, is a *non-abelian* group.

Problem 4. (20 pts) In this problem we explore basic compositions of rotations and translations. Solve the following parts:

(a) Let $\theta, \phi \in S^1$ be two angles. Show that

$$R_\theta \circ R_\phi = R_{\theta+\phi}.$$

(b) Let $\theta \in S^1$ be an angle. Find the unique angle $\phi \in S^1$ such that $R_\theta \circ R_\phi = \mathrm{Id}$ is the identity map $\mathrm{Id}(x, y) = (x, y)$.

(c) Let (α, β) and (γ, δ) be two points in the Euclidean Plane $\mathbb{R}^2$. Prove that

$$t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)}.$$

(d) Let $(\alpha, \beta) \in \mathbb{R}^2$ be a point in Euclidean plane. Find the unique $(\gamma, \delta) \in \mathbb{R}^2$ such that the composition $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = \mathrm{Id}$.

Solution.

(a) Rather than use function notation, it is more convenient to identify a linear operator (such as a rotation) with its corresponding matrix in the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. Then map composition is matrix multiplication, and we find

$$\begin{aligned} R_\theta \circ R_\phi &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \\ &= R_{\theta+\phi}, \end{aligned}$$

where we used the famous formulas for the sine and cosine of the sum of two angles. This formalizes the idea that rotating twice should be the same as rotating once through the angle equal to the sum of the angles of the two individual rotations.

Remark: Notice that, in addition, we have

$$R_\theta \circ R_\phi = R_{\theta+\phi} = R_{\phi+\theta} = R_\phi \circ R_\theta,$$

so rotations commute.

(b) The identity map corresponds to the matrix

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so we are searching for an angle ϕ such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{Id} = R_\theta \circ R_\phi = R_{\theta+\phi} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$$

This will be satisfied if and only if

$$\cos(\theta+\phi) = 1 \qquad \text{and} \qquad \sin(\theta+\phi) = 0.$$

The only angle with cosine 1 and sine 0 is the angle 0, so $\theta + \phi = 0$, meaning $\phi = -\theta$, also showing uniqueness. Therefore, rotations have unique inverses given by rotations through the opposite angle.

Remark: In terms of algebra, we say that rotations of the plane form an abelian group, $\mathrm{SO}(2,\mathbb{R})$. This is how the circle S^1 is given the structure of a group.

(c) Function notation is now preferred, because translations are not linear maps. For any point $(x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned} t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)}(x,y) &= t_{(\alpha,\beta)}(t_{(\gamma,\delta)}(x,y)) \\ &= t_{(\alpha,\beta)}(x+\gamma, y+\delta) \\ &= (x+\gamma+\alpha, y+\delta+\beta) \\ &= t_{(\alpha+\gamma,\beta+\delta)}(x,y). \end{aligned}$$

Since the identity holds for all inputs, we conclude that the functions are equal, $t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)}$. As with rotations, translating twice along two vectors is the same as translating once along the sum of those vectors.

Remark: Again notice the commutativity property:

$$t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)} = t_{(\alpha+\gamma,\beta+\delta)} = t_{(\gamma+\alpha,\delta+\beta)} = t_{(\gamma,\delta)} \circ t_{(\alpha,\beta)}.$$

(d) We want to find a vector $(\gamma, \delta) \in \mathbb{R}^2$ such that, for all points $(x, y) \in \mathbb{R}^2$, we have

$$\mathrm{Id}(x, y) = t_{(\alpha,\beta)} \circ t_{(\gamma,\delta)}(x, y).$$

Evaluating both sides, we have

$$(x, y) = (x + \gamma + \alpha, y + \delta + \beta).$$

Equal points must have equal coordinates, so this means

$$x = x + \gamma + \alpha \qquad \text{and} \qquad y = y + \delta + \beta,$$

which solves to $\gamma = -\alpha$ and $\delta = -\beta$, so the desired vector is $(\gamma, \delta) = (-\alpha, -\beta)$, and it is unique.

Remark: In terms of algebra, we say that translations of the plane form an abelian group. This group is isomorphic to $\mathbb{R}^2$ itself (under point wise addition), because we can always identify a translation $t_{(\alpha,\beta)}$ with the point (α, β) to which it sends the origin.

Problem 5. (20 pts) Let $L = \{(x, y) : y = 0\} \subseteq \mathbb{R}^2$ and $M = \{(x, y) : x = 0\} \subseteq \mathbb{R}^2$ be the x and y-axis respectively.

(a) Show that $\overline{r}_L\overline{r}_M(x, y) = (-x, -y)$.

(b) Prove that there exists no line $N \subseteq \mathbb{R}^2$ such that

$$\overline{r}_N = \overline{r}_L\overline{r}_M,$$

where L, M are as in Part (a). Thus, we learn that the composition of reflections is *not* always a reflection.

(c) Find an angle $\phi \in S^1$ such that the composition $\overline{r}_L\overline{r}_M$ in Part (a) equals the rotation R_ϕ, i.e. $R_\phi = \overline{r}_L\overline{r}_M$. Thus, we learn that the composition of reflections can *sometimes* be a rotation.

(d) Find all the angles $\theta \in S^1$, if any, such that the rotation R_θ, centered at the origin, *commutes* with any reflection $\overline{r}_L$, where L is a line through the origin:

$$R_\theta \circ \overline{r}_L = \overline{r}_L \circ R_\theta.$$

Solution.

(a) We already know that reflection in L is the standard reflection, $\overline{r}_L(x,y) = (x,-y)$. By drawing a picture or constructing the conjugation, we can see that $\overline{r}_M(x,y) = (-x,y)$. Then

$$\overline{r}_L\overline{r}_M(x,y) = \overline{r}_L(-x,y) = (-x,-y)$$

for any $(x,y) \in \mathbb{R}^2$.

Remark: Note that you get the same result if you compute $\overline{r}_M\overline{r}_L(x,y)$, so these reflections *commute*.

(b) Notice that $\overline{r}_L\overline{r}_M$ is rotation about the origin through and angle π. Therefore, it fixes only the origin. That is, for $(x,y) \in \mathbb{R}^2$,

$$\overline{r}_L\overline{r}_M(x,y) = (-x,-y)$$

is equal to (x,y) if and only if $(x,y) = (0,0)$. But if $\overline{r}_L\overline{r}_M = \overline{r}_N$ for some line N, then $\overline{r}_L\overline{r}_M$ would fix every point on N. Since $\overline{r}_L\overline{r}_M$ fixes only the origin, no such line N can exist.

Remark: Any composition of an *even* number of reflections is never a reflection. Heuristically, reflections reverse the orientation of the plane, so an even number of them in a row will preserve orientation. As another extreme example, for any reflection $\overline{r}_S$, we have $\overline{r}_S\overline{r}_S = \mathrm{Id}$. But the identity is not a reflection, because reflections fix only a single line, while the identity fixes the whole plane.

(c) As remarked in the solution to (b), $\phi = \pi$ will suffice. To see why, notice that that

$$R_\pi(x,y) = (x\cos\pi - y\sin\pi, x\sin\pi + y\cos\pi) = (-x,-y) = \overline{r}_L\overline{r}_M(x,y).$$

Since these two functions agree on all points $(x,y) \in \mathbb{R}^2$, they are the same function.

(d) The only possible angles are 0 and π. If we are looking at lines L through the origin, then reflections through these lines (with the exception of the one vertical line through the origin) are given by Eq. (1) for various values of m, setting $b = 0$. These formulas are then linear (no constants or higher powers of x or y). The remaining vertical reflection is also linear, because it is the map $(x,y) \mapsto (-x,y)$ from part (a). (Alternatively, one can see that these are all linear because they are conjugations of the standard reflection by rotations through the origin.) Therefore, we can treat all equations like matrix equations (this will allow us to pull in minus signs below).

Clearly the angle $0 \in S^1$ satisfies this condition, because $R_0 = \mathrm{Id}$, and so

$$R_0\overline{r}_L = \mathrm{Id}\,\overline{r}_L = \overline{r}_L = \overline{r}_L\,\mathrm{Id} = \overline{r}_L R_0$$

for any line L through the origin. For the angle $\theta = \pi$, we notice from our calculation in part (c) that $R_\pi = -\operatorname{Id}$, so a nearly identical calculation gives

$$R_\pi \overline{r}_L = -\operatorname{Id}\overline{r}_L = -\overline{r}_L = -\overline{r}_L \operatorname{Id} = \overline{r}_L(-\operatorname{Id}) = \overline{r}_L R_\pi$$

for any line L through the origin.

No other angle is possible. Fix an angle $\theta \in S^1$ with $\theta \neq 0$ and $\theta \neq \pi$. We need to produce a line L through the origin such that

$$R_\theta \overline{r}_L \neq \overline{r}_L R_\theta.$$

In fact, the commutativity condition will fail for *all* such lines L. So let L be any line through the origin. To show that our condition fails, we just need to show that the functions $R_\theta \overline{r}_L$ and $\overline{r}_L R_\theta$ take different values on at least one point P. Let P be a point on the line L different from the origin. Then it is fixed by the reflection $\overline{r}_L$, so we have

$$R_\theta \overline{r}_L(P) = R_\theta(P).$$

On the other hand, we need to look at

$$\overline{r}_L R_\theta(P).$$

If we had the equality $R_\theta(P) = \overline{r}_L R_\theta(P)$, then that means that the point $R_\theta(P)$ is fixed by the reflection in L, so $R_\theta(P)$ must lie on the line L. But R_θ is an isometry, so $R_\theta(P)$ is the same distance from 0 as is P. There are only two such points on L, namely P and $-P$. Since P is not the origin, it is not fixed by R_θ (remember $\theta \neq 0$, so R_θ fixes only the origin), so we know that $R_\theta(P) \neq P$. Therefore, $R_\theta(P) = -P$. But no two rotations send P to the same location (why?), and $R_\pi(P) = -P$ already, so we would be forced to conclude that $\theta = \pi$. Since we are assuming $\theta \neq \pi$, we have a contradiction, meaning that

$$R_\theta \overline{r}_L(P) \neq \overline{r}_L R_\theta(P),$$

so the functions cannot be equal.

In conclusion, rotations through 0 and π commute with *every* reflection in a line through the origin, and other rotations commute with *none* of these reflections.

Problem 6. (20 pts) Consider the square $S \subseteq \mathbb{R}^2$ with four vertices given by the points $(-1,-1), (-1,1), (1,1), (1,-1) \in \mathbb{R}^2$. The square consists of the convex hull of these four points, i.e. the region given by

$$S = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

(a) Show that there exists *no* translation $t_{(\alpha,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, except for the identity, such that $t_{(\alpha,\beta)}(S) \subseteq S$, i.e. the translation sends the square to the square.

(b) Find *four* distinct lines $L_1, L_2, L_3, L_4 \subseteq \mathbb{R}^2$ such that the reflections $\overline{r}_{L_i}$, $1 \leq i \leq 4$, all satisfy the inclusion $\overline{r}_{L_i}(S) \subseteq S$.

(c) Find 27 distinct lines $M \subseteq \mathbb{R}^2$ such that $\overline{r}_M(S) \not\subseteq S$, i.e. the reflection $\overline{r}_M$ maps the square S *not inside* the square S.

(d) Find all angles $\theta \in S^1$ such that $R_\theta(S) \subseteq S$.

(e) Find *infinitely many* angles $\theta \in S^1$ such that $R_\theta(S) \not\subseteq S$.

Solution.

(a) Suppose $t_{(\alpha,\beta)}$ is such a translation. We show that it maps at least one corner of S to a point not in S, unless $(\alpha, \beta) = (0, 0)$. If either $\alpha < 0$ or $\beta < 0$ then $t_{(\alpha,\beta)}(-1, -1) = (-1 + \alpha, -1 + \beta)$ has a coordinate less than -1, so it is not in S. If either $\alpha > 0$ or $\beta > 0$ then $t_{(\alpha,\beta)}(1, 1) = (1 + \alpha, 1 + \beta)$ has a coordinate greater than 1, so it is not in S. Therefore, neither of these are possible if we require that $t_{(\alpha,\beta)}(S) \subseteq S$, so we are forced to conclude $\alpha = \beta = 0$. This means that our translation is

$$t_{(\alpha,\beta)} = t_{(0,0)} = \mathrm{Id},$$

the identity map.

(b) Our four lines are the x- and y-axes along with the lines through the origin with slopes 1 and -1. They all map S to S exactly, which you can see by drawing them on a sheet of paper and folding along each line.

(c) Any lines other than those described in (b) will suffice. Setting b to 0 in Eq. (1) gives linear equations for reflections in all of these lines. That formula (with $b = 0$) is identical to the action of the matrix

$$\frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

The image of this matrix on the corner $(1, 1) \in S$ is

$$\overline{r}_m(1,1) = \frac{1}{m^2+1}\begin{bmatrix} -m^2+2m+1 \\ m^2+2m-1 \end{bmatrix},$$

Where $\overline{r}_m$ denotes reflection through the line $y = mx$. We have already covered the cases $m = 0$, $m = 1$ and $m = -1$ (the vertical line can be thought of as $m = \infty$ or $m = -\infty$).

If $m < -1$, then the x-component of $\overline{r}_m(1, 1)$ is less than -1, which can be checked by multiplying by the denominator and simplifying. This means that that $\overline{r}_m(1, 1)$ is not in the square S. Similarly, if $-1 < m < 0$, then the y-component of $\overline{r}_m(1, 1)$ is less than -1. Next, if $0 < m < 1$, then the x-component of $\overline{r}_m(1, 1)$ is greater than 1. Finally, if $1 < m$, then the y-component of $\overline{r}_m(1, 1)$ is greater than 1, so $\overline{r}_m(1, 1)$ is never in the square S if $m \neq 0$, $m \neq 1$, and $m \neq -1$.

Therefore, for any line M not considered in part (b), $\overline{r}_M(S) \not\subseteq S$. Pick your 27 favorite slopes. I like $m = \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{27}}$.

(d) By drawing congruent squares on two sheets of paper and turning the paper over the second, we can immediately see that the angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ map the square to the square exactly. We show in part (e) that these are the only such angles.

(e) The image of the corner $(1,1)$ under the rotation R_θ is

$$R_\theta(1,1) = (\cos\theta - \sin\theta, \cos\theta + \sin\theta) = \left(\sqrt{2}\sin\left(-\theta + \frac{\pi}{4}\right), \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)\right).$$

For $0 < \theta < \frac{\pi}{2}$, the y-component of $R_\theta(1,1)$ is greater than 1 (check this using the unit circle), so $R_\theta(1,1)$ is not in the square S. Similarly, for $\frac{\pi}{2} < \theta < \pi$, the x-component of $R_\theta(1,1)$ is less than 1. Next, for $\pi < \theta < \frac{3\pi}{2}$, the y-component of $R_\theta(1,1)$ is less than 1. Finally, for $\frac{3\pi}{2} < \theta < 2\pi$, the x-component of $R_\theta(1,1)$ is greater than 1, so $R_\theta(1,1)$ is never in the square S if θ is none of 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Therefore, for any angle $\theta \in S^1$ not considered in part (d), we have $R_\theta(S) \not\subseteq S$.

Problem 7. (20 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) The linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is an isometry.

(b) Any linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with $a \neq 0$, must be an isometry.

(c) The composition $f \circ g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of two isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is always an isometry.

(d) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry. If there exist infinitely many points $P \in \mathbb{R}^2$ such $f(P) = P$, then $f = \mathrm{Id}$ must be the identity.

(e) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isometry which fixes the points $(0,0)$, $(1,0)$ and $(0,1)$, i.e. $f(0,0) = (0,0)$, $f(1,0) = (1,0)$ and $f(0,1) = (0,1)$. Then $f = \mathrm{Id}$ must be the identity.

(f) The composition of reflections is *always* a reflection.

(g) The composition of rotations centered at the origin are *always* rotations.

(h) The composition of translations is *always* a translation.

(i) There is an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that sends the square S, as defined in Problem 6, strictly inside itself, i.e.

$$f(S) \subsetneq \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\}.$$

(j) For any rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, there exists a power $n \in \mathbb{N}$ such that the composition $R_\theta^n = \mathrm{Id}$.

Solution.

(a) True. This is the matrix for $R_{\pi/2}$, clockwise rotation about the origin through $\frac{\pi}{2}$. We have seen that all rotations are isometries.

(b) False. We saw in Discussion 1 the case where $a = 1$, a shear, and we showed that it is not an isometry. In fact, this map can *never* be an isometry, no matter the value of $a \neq 0$. This is because $d((0, 1), (0, 0)) = 1$, while

$$d((f(0, 1), f(0, 0)) = d((a, 1), (0, 0)) = \sqrt{a^2 + 1} \neq 1$$

if $a \neq 0$.

(c) True. This is part of why isometries form a group. To prove this, we need to show that the composition $f \circ g$ preserves the distance between points, meaning that for any two points $P, Q \in \mathbb{R}^2$, we have

$$d(f \circ g(P), f \circ g(Q)) = d(P, Q).$$

Since we are told that g is an isometry, we already know that

$$d(g(P), g(Q)) = d(P, Q).$$

Finally, since f is an isometry, it also preserves the distance between any points, in particular $g(P)$ and $g(Q)$, so

$$d(f \circ g(P), f \circ g(Q)) = d(f(g(P)), f(g(Q)) = d(g(P), g(Q) = d(P, Q).$$

We conclude that $f \circ g$ is an isometry of the plane.

(d) False. Any reflection through a line preserves all points on that line, and every line contains an infinite number of points. In particular, in Problem 3(b), we checked directly that the reflection $\overline{r}_L$ preserves all points on L, meaning that if P is on L, we have $\overline{r}_L(P) = P$. And this isometry is certainly not the identity.

(e) True. Recall that a linear map is determined uniquely by its action on a basis. Since we are given that f is linear, and we know what it does to the basis $\{(1, 0), (0, 1)\}$, it must be determined uniquely. Since the identity Id achieves all three data points that we care about, we must have $f = \mathrm{Id}$. In fact, given a basis, the columns of the matrix of a linear map are exactly the images of the basis elements, so since $f(1,0) = (1,0)$ and $f(0,1) = (0,1)$, we can see that the matrix for f must have first column $\begin{bmatrix}1\\0\end{bmatrix}$ and second column $\begin{bmatrix}0\\1\end{bmatrix}$, so it is the identity matrix.

Remark: Notice that we did not need the data point $f(0,0) = (0,0)$ to conclude that f was the unique *linear* isometry satisfying the requirements. Without using this data point, can we conclude that f is the unique isometry overall? No, because in addition to the identity, there is another isometry which fixes $(1,0)$ and $(0,1)$ (reflection in the line connecting these points).

However, if we consider all three fixed points, f is indeed uniquely determined to be the identity (among all isometries, linear or not). The Lemma at the beginning of Stillwell 1.4 (p. 9) guarantees that there is at most one isometry that can send three non-collinear points A B, C to three determined locations $f(A)$, $f(B)$, and $f(C)$. Clearly $(0,0)$, $(1,0)$, and $(0,1)$ are not on a line, and Id fixes each of them as desired. Then by the Lemma, this can be the only isometry which does so.

(f) False. This is the result of Problem 5(b).

(g) True. We proved this in Problem 4(a).

(h) True. We proved this in Problem 4(d).

(i) False. Suppose f is such an isometry. Then

$$d(f(1,1), f(-1,1)) = d((1,1),(-1,-1)) = 2\sqrt{2}.$$

Let $P = f(1,1)$ and $Q = f(-1,-1)$. Then $P, Q \in f(S)$ are points in the image of the square, and they are a distance $2\sqrt{2}$ from each other. The only pairs of points in S which achieve this distance are $(1,1)$ with $(-1,-1)$, and $(1,-1)$ with $(-1,1)$. But none of these four points are in $f(S)$, so none of them can be P or Q. We've arrived at a contradiction, so f cannot be an isometry.

(j) False. This is a fun one. By induction on the result of Problem 4(a), we can show that for any angle $\theta \in S^1$, we have

$$R_\theta^n = R_{n\theta}.$$

In order for this rotation to be the identity, we would need $n\theta$ to be a multiple of 2π, so the question becomes: for any angle $\theta \in S^1$, is there a multiple of θ which is also some multiple of 2π?

The answer is no. Take θ to be an irrational multiple of 2π, meaning $\theta = 2\pi\alpha$ for some irrational number $\alpha \in \mathbb{R}$. We are searching for an integer n such that

$$n\theta = 2\pi k$$

for some integer k. Plugging in $\theta = 2\pi\alpha$ and rearranging, we have

$$\alpha = \frac{k}{n}$$

which has no solution for integers n and k, because 2π is irrational. Since there does not exist a multiple $n\theta$ which is a multiple of 2π, we conclude that no power of R_θ will ever be the identity.

Remark: Notice that, given an angle $\theta \in [0, 2\pi)$, the ratio $\alpha = \theta/2\pi$ is in the interval $[0, 1)$. This ratio determines whether or not there exists a power of R_θ equal to Id. If α is irrational, the above proof shows that such a power is never possible (no matter how many times you rotate around by θ, you'll never get back to where you started). If α is rational, represent it as $\alpha = \frac{p}{q}$ for integers p, q in lowest terms. Then

$$q\theta = (q\alpha)2\pi = p\,2\pi$$

is a multiple of 2π, so $R_\theta^q = \text{Id}$. In an analytic sense, almost all angles are irrational multiples of 2π, so if you pick an angle θ at random, there will never be a power of R_θ equal to the identity.

However, if θ is an irrational multiple of 2π, by taking higher and higher powers of R_θ, you can always get *arbitrarily close* to Id (though never exactly Id). In fact, you can always get arbitrarily close to *any* rotation R_ϕ about the origin, for any ϕ. This is known as an ergodic theorem; it says that the orbit of multiples of θ is uniformly distributed around the circle. This does not happen if θ is a rational multiple of 2π (why?).