MAT 141: PROBLEM SET 4

DUE TO FRIDAY FEB 21 AT 10:00AM

Abstract. This is the fourth problem set for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. It was posted online on Saturday Feb 15 and is due Friday Feb 21 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice problems on spherical geometry on the 2-sphere S^2. In particular, we would like to become familiar with the distance d_{S^2} in the 2-sphere, its group of isometries and the notion of a line in S^2.

Task: Solve Problems 1 through 7 below. The first and last problems will not be graded but I trust that you will work on them. Problems 2 to 6 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Decide whether the following points $P \in \mathbb{R}^3$ belong to the 2-sphere S^2:

$$(1,0,0),(0,1,0),(1,1,0),(1/\sqrt{2},1/\sqrt{2},0),$$

$$(1/2,0,1/2),(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}),(1/4,1/2,1/2).$$

Problem 2. (20 pts) For each pair of axis $l_1, l_2 \subseteq \mathbb{R}^3$, find a *linear* isometry $\varphi \in \text{Iso}(\mathbb{R}^3)$, $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, such that $\varphi(l_1) = l_2$.

(a) Let $l_1 = \langle(1,0,0)\rangle$ be the oriented axis spanned by the vector $(1,0,0)$, and choose $l_2 = \langle(0,0,1)\rangle$.

(b) Let $l_1 = \langle(0,1,0)\rangle$ and choose $l_2 = \langle(0,0,1)\rangle$.

(c) Let $l_1 = \langle(1,1,0)\rangle$ and choose $l_2 = \langle(0,0,1)\rangle$.

(d) Let $l_1 = \langle(1,1,2)\rangle$ and choose $l_2 = \langle(0,0,1)\rangle$.

Problem 3. (20 pts) **Distances in S^2**. Let (S^2, d_{S^2}) be the 2-sphere with its distance function d_{S^2}. We consider the set $S^2 \subseteq \mathbb{R}^2$ as the set of points

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and thus use Cartesian coordinates $(x, y, z) \in S^2$ for points in S^2.

(a) Compute the distance $d_{S^2}(P, Q)$ between $P = (0, 0, 1)$ and $Q = (0, 1, 0)$.

(b) Compute the distance between $(1/\sqrt{2}, 1/\sqrt{2}, 0) \in S^2$ and $P = (0, 0, 1)$.

(c) Draw the following four sets in the S^2:

$$E_{\pi/4} = \{R \in S^2 : d_{S^2}(P, R) = \pi/4\}, \quad E_{\pi/2} = \{R \in S^2 : d_{S^2}(P, R) = \pi/2\},$$

$$E_{3\pi/4} = \{R \in S^2 : d_{S^2}(P, R) = 3\pi/4\}, \quad E_{\pi} = \{R \in S^2 : d_{S^2}(P, R) = \pi\}.$$

(d) Show that given any two points $P_1, P_2 \in \mathbb{S}^2$, we have the equality

$$\{R \in S^2 : d_{S^2}(P_1, R) = d_{S^2}(P_2, R)\} = \{R \in \mathbb{R}^3 : d_{\mathbb{R}^3}(P_1, R) = d_{\mathbb{R}^3}(P_2, R)\} \cap S^2,$$

comparing the sets of equidistant points R to P_1, P_2 in S^2 and $\mathbb{R}^3$.

Problem 4. (20 pts) **Lines in S^2**. Let $L, M \subseteq S^2$ be two distinct lines. The map $a : S^2 \longrightarrow S^2$ defined by $(x, y, z) \longmapsto (-x, -y, -z)$ is called the antipodal map.

(a) Show that $L \cap M$ consists of exactly two distinct points P, Q.

(b) Let $L \cap M = \{P, Q\}$, show that $Q = a(P)$ and $P = a(Q)$, where $a : S^2 \longrightarrow S^2$ is the antipodal map.

(c) Show that for any line $L \subseteq S^2$ there exists a plane $\Pi_L \subseteq \mathbb{R}^3$ through the origin such that $L = \Pi_L \cap S^2$.

(d) Let $\widehat{\Pi} \subseteq \mathbb{R}^3$ be a 2-plane which does *not* contain the origin and such that the intersection $\widehat{\Pi} \cap S^2$ contains more than a point. Show that $\widehat{\Pi} \cap S^2$ must be a circle.

(e) In the same hypothesis of Part.(d), show that $\widehat{\Pi} \cap S^2$ is not a line in S^2.

Problem 5. (20 pts) **The antipodal map in S^2**. Consider the antipodal map $a : S^2 \longrightarrow S^2$ defined by $(x, y, z) \longmapsto (-x, -y, -z)$.

(a) Show that a is an isometry of (S^2, d_{S^2}) and show that is has no fixed points.

(b) Express a as a composition of reflections in $\text{Iso}(S^2)$.

(c) Let $f \in \text{Iso}(S^2)$ be an isometry. Show that antipodal points P, Q remain antipodal after applying $f : S^2 \longrightarrow S^2$, i.e. prove that P, Q are antipodal if and only $f(P), f(Q)$ are antipodal.

Problem 6. (20 pts) **Isometries in** S^2. Let $l \subseteq \mathbb{R}^3$ be the oriented axis in 3-space generated by the vector $v = (1, 1, 1)$.

(a) Express the rotation $R_{z,\pi/2}$ as a composition of two reflections.

(b) Where does $R_{z,\pi/2}$ send the point $\frac{1}{\sqrt{6}}(1, 1, 2) \in S^2$?

(b) Find a formula for the rotation $R_{l,\theta}$.

(c) Where does $R_{l,\theta}$ map the point $(0, 0, 1)$?

(d) Let $A \subseteq \mathbb{R}^3$ be any oriented axis. Show that a general rotation $R_{A,\theta}$ must have exactly two fixed points.

(e) In the hypothesis of Part (d), show that the two fixed points of a general rotation $R_{A,\theta}$ must be antipodal.

Problem 7. **Real-Life Computation**. Consider the longitude φ (azimuth angle) and latitude θ coordinates on Earth. Suppose the surface of the Earth is spherical, its core is at $(0, 0, 0) \in \mathbb{R}^3$, and the radius of the Earth is $r = 6378$ kilometers.

In this coordinates, the point $(x, y, z) \in \mathbb{R}^3$ corresponds to

$$(x, y, z) = (r \cos\theta \sin\varphi, r \cos\theta \cos\varphi, r \sin\theta),$$

with $\theta \in [-\pi/2, \pi/2], \varphi \in [-\pi, \pi)$, where $\theta = \pi/2$ is the North Pole and $\theta = -\pi/2$ is the South Pole, and $\varphi \in (0, \pi)$ is East of the Greenwich Meridian, and $\varphi \in [-\pi, 0)$ is West of the Greenwich Meridian.

(a) UC Davis is located at $(\theta, \varphi) = (38.5382°N, 121.7617°W)$ and Barcelona (Spain) at $(\theta, \varphi) = (41.3851°N, 2.1734°E)$. Compute approximately the distance d_{S^2} on the surface of Earth from UC Davis to Barcelona.

(b) UC Berkeley is located at $(\theta, \varphi) = (37.8719°N, 122.2585°W)$. Compute the distance d_{S^2} on the surface of Earth from UC Davis to UC Berkeley.

(c) Compare the distances on the surface of Earth with the corresponding distances considered in $\mathbb{R}^3$. In which case is the distance $d_{\mathbb{R}^3}$ closer to the distance d_{S^2} on the surface of Earth ?