MAT 141: PROBLEM SET 5

DUE TO FRIDAY FEB 28 AT 10:00AM

Abstract. This is the fifth problem set for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. It was posted online on Friday Feb 21 and is due Friday Feb 28 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice problems on spherical geometry on the 2-sphere S^2. In particular, we would like to become familiar with manipulating lines in the 2-sphere, and study polygonal regions within S^2.

Task: Solve Problems 1 through 7 below. The first two problems will not be graded but I trust that you will work on them. **Problems 3 to 7 will be graded**.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Show that there are *no* parallel lines $L_1, L_2 \subseteq S^2$ in the 2-sphere, i.e. if $L_1, L_2 \subseteq S^2$ are lines, then $L_1 \cap L_2$ is non-empty.

Problem 2. **Triangles in the Euclidean Plane.** Let $T \subseteq \mathbb{R}^2$ be a triangle, and α, β, γ be the interior angles of T.

(a) Show that $\alpha + \beta + \gamma = \pi$.

(b) Construct a triangle T' with the same interior angles α, β, γ, such that
$$Area(T') = 293 \cdot Area(T).$$

(c) Prove that it is not possible to compute the area of a triangle $T \subseteq \mathbb{R}^2$ just by knowing its interior angles α, β, γ.

Problem 3. (20 pts) **Triangles in Euclidean Surfaces.** In this problem C is the Euclidean cylinder, M is the twisted Euclidean cylinder, T^2 is the Euclidean 2-torus, and K is the Klein bottle. We will denote an arbitrary Euclidean surface by S.

(a) Give two triangles $T_1, T_2 \subseteq C$ such that the interior angles of T_1 coincide with the interior angles of T_2, but the area of T_1 is distinct from the area of T_2.

(b) Do there exist two triangles $T_1, T_2 \subseteq T^2$ with the same interior angles but different area ? How about $T_1, T_2 \subseteq K$ in the Klein bottle ?

(c) Let $T \subseteq S$ be a triangle with sides given by the lines $L_1, L_2, L_3 \subseteq S$. How many regions does the complement $S \setminus \{L_1, L_2, L_3\}$ has ?

Answer for each of the five cases $S = \mathbb{R}^2, S = C, S = M, S = T^2$ and $S = K$.

(d) Let $T \subseteq S$ be a triangle in an arbitrary Euclidean surface S, and α, β, γ be the interior angles of T. Show that $\alpha + \beta + \gamma = \pi$.

Problem 4. (20 pts) **Triangles in S^2 (Part I)**. Let $T \subseteq S^2$ be a triangle in S^2, defined by the lines $L_1, L_2, L_3 \subseteq S^2$.

(a) Show that the area of the unit radius 2-sphere is 4π.

(b) Show that the area of a sector of angle α is $\alpha/2\pi$ the area of the 2-sphere. A *sector* of angle α is the bigon described by two lines at angle α.

(c) Show that the complement $S^2 \setminus \{L_1, L_2, L_3\}$ consists of eight *triangular* regions, where $L_1, L_2, L_3 \subseteq S^2$ are lines defining a triangle.

(d) Show that there exist six pairs of such regions such that the union of the pair of region is a sector as in Part (b).

(e) Let $T \subseteq S^2$ be a triangle, and α, β, γ be the interior angles of T. Show that

$$\alpha + \beta + \gamma = \pi + Area(T).$$

Problem 5. (20 pts) **Triangles in S^2 (Part II)**.

(a) Show that there exists a triangle $T \subseteq S^2$ such that all its interior angles α, β, γ are distinct from $\pi/2$.

(b) Let $\varepsilon \in \mathbb{R}^+$, prove that there exist triangles $T \subseteq S^2$ such that *one* of its interior angles is less than ε. That is, there are triangles with one of its angles arbitrarily small.

(c) Let $\varepsilon \in \mathbb{R}^+$, prove that there exist triangles $T \subseteq S^2$ such that *two* of its interior angles are less than ε. That is, there are triangles with two of its angles arbitrarily small.

(d) Let $\varepsilon \in \mathbb{R}^+$ be given. Construct a triangle $T \subseteq S^2$ whose area is $2\pi - \varepsilon$, i.e. build a triangle whose are is arbitrarily close to the area of half the 2-sphere S^2.

(e) Is is possible to have triangles $T \subseteq S^2$ with arbitrarily small area ?

Problem 6. (20 pts) **Polygons in $\mathbb{R}^2$ and** S^2. Let P be an n-sided polygon, i.e. a region of S^2 bounded by n lines $L_1, L_2, \ldots, L_n$. For instance, $n = 3$ is a triangle.

(a) Let $P \subseteq \mathbb{R}^2$ be a polygon *in the plane*, and $\alpha_1, \ldots, \alpha_n$ its interior angles. Find a formula for the sum $\alpha_1 + \ldots + \alpha_n$ in terms of n.

(b) Find two polygons $P_1, P_2 \subseteq \mathbb{R}^2$ with the same interior angles but different areas.

(c) Let $P \subseteq S^2$ be a polygon *in the sphere*, and $\alpha_1, \ldots, \alpha_n$ its interior angles. Find a formula for the sum $\alpha_1 + \ldots + \alpha_n$ in terms of n and $Area(P)$.

(d) Show that two polygons $P_1, P_2 \subseteq S^2$ with the same interior angles must have the same area.

Problem 7. (20 pts) **Tilings of the 2-sphere** S^2. In this problem we will study how to regularly tile the 2-sphere, i.e. how to regularly cover the 2-sphere with polygons of the same area.

(a) Show that S^2 can be subdivided into four triangles of the same area.

(b) Show that S^2 can be subdivided into eight triangles of the same area.

(c) Show that S^2 can be subdivided into twenty triangles of the same area.

(d) Show that S^2 can be subdivided into six squares of the same area.

(e) Show that S^2 can be subdivided into twelve pentagons of the same area.

(f) (**Challenging**) Show that if S^2 is subdivided into n-gons, all with the same area and same length of sides, then the subdivision must be as into triangles, squares or pentagons.

(Part (f) will not be graded.)