MAT 141: PROBLEM SET 6

DUE TO FRIDAY MAR 6 AT 10:00AM

Abstract. This is the sixth problem set for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. It was posted online on Saturday Feb 29 and is due Friday March 6 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice problems on spherical geometry on the 2-sphere S^2. In particular, we would like to become familiar with the *stereographic projection* manipulating lines in the 2-sphere, and study polygonal regions within S^2.

Task: Solve Problems 1 through 5 below. **Problems 1 to 5 will be graded**.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. (20 pts) **Regions in Euclidean Plane and 2-Sphere.** A set of lines $\mathcal{L} := \{L_1, L_2, \dots, L_n\}$ be a set of n lines is said to be *generic* if not two lines in $\mathcal{L}$ are parallel, and no three lines in $\mathcal{L}$ intersect at a point.

(a) Let $\mathcal{L} = \{L_1, L_2, \dots, L_n\}$ be a generic set of n lines in $\mathbb{R}^2$. Show that the complement
$$\mathbb{R}^2 \setminus (L_1 \cup L_2 \cup \ldots \cup L_n),$$
of the lines in $\mathcal{L}$ inside the Euclidean plane $\mathbb{R}^2$ consists of $\dfrac{n^2+n+2}{2}$ regions.

(b) Show that Part.(a) might *not* hold if $\mathcal{L}$ is not necessarily a generic set.

(c) Let $\mathcal{L} = \{L_1, L_2, \dots, L_n\}$ be a generic set of n lines in S^2. Show that the complement
$$S^2 \setminus (L_1 \cup L_2 \cup \ldots \cup L_n),$$
of the lines in $\mathcal{L}$ inside the Euclidean plane S^2 consists of $n^2 - n + 2$ regions.

(d) Show that Part.(c) might *not* hold if $\mathcal{L}$ is not necessarily a generic set.

(e) Consider $\mathcal{L}$ as in Part.(c). Show that the *total* number of intersections of all the lines in $\mathcal{L}$ is $n(n-1)$.

Problem 2. (20 pts) **Stereographic Projection From North and South Poles.** Let $N = (0,0,1) \in S^2$ be the North pole and $S = (0,0,-1)$ the South pole of the 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Let us identify the 2-plane $\Pi = \{(x,y,z) \in \mathbb{R}^3 : z = 0\} \subseteq \mathbb{R}^3$ with the Euclidean plane $\mathbb{R}^2 = \{(u,v) : u,v \in \mathbb{R}^2\}$ via $(x,y,0) = (u,v)$.

(a) Let $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ be the stereographic projection from the North pole N, defined by
$$\pi_N(P) = Q, \quad P \in S^2, Q \in \mathbb{R}^2,$$
where Q is the unique intersection point $L_{P,N} \cap \Pi$ distinct from N, and $L_{P,N} \subseteq \mathbb{R}^3$ is the unique line containing $P, N \in \mathbb{R}^3$. Find a formula of $\pi_N(P) = (u,v)$ in terms of $P = (x,y,z)$.

(b) Find a formula for the inverse map $\pi_N^{-1} : \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\}$, i.e. the unique map such that $\pi_N^{-1} \circ \pi_N = id_{S^2}$ and $\pi_N \circ \pi_N^{-1} = id_{\mathbb{R}^2}$.

(c) Let $\pi_S : S^2 \setminus \{S\} \longrightarrow \mathbb{R}^2$ be the stereographic projection from the South pole S, defined by
$$\pi_S(P) = Q, \quad P \in S^2, Q \in \mathbb{R}^2,$$
where Q is the unique intersection point $L_{P,S} \cap \Pi$ distinct from S, and $L_{P,S} \subseteq \mathbb{R}^3$ is the unique line containing $P, S \in \mathbb{R}^3$. Find a formula of $\pi_S(P) = (u,v)$ in terms of $P = (x,y,z)$.

(d) Show that there exists an isometry $\varphi : S^2 \longrightarrow S^2$ such that $\pi_S = \pi_N \circ \varphi$. Is this isometry unique ?

Problem 3. (20 pts) **Properties of π_N.** Let $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ be the stereographic projection from the North pole.

(a) Show that π_N is bijective.

(b) Let $P \in S^2$ a point in the equator $E = S^2 \cap \Pi$. Show that $\pi_N(x,y,z) = (x,y)$.

(c) Describe the image of the lower hemisphere $S^2 \cap \{(x,y,z) : z \leq 0\}$.

(d) Describe the image of the upper hemisphere $(S^2 \setminus \{N\}) \cap \{(x,y,z) : z \geq 0\}$.

(e) Let $\theta \in S^1$ be an angle. Find an isometry $\phi \in \mathrm{Iso}(S^2 \setminus \{N\})$ such that
$$\pi_N \circ \phi = R_{(0,0),\theta} \circ \pi_N,$$
where $R_{(0,0),\theta}$ in the rotation of angle θ centered at the origin $(0,0) \in \mathbb{R}^2_{u,v}$.

Problem 4. (20 pts) **Stereographic Projection and Lines.** Let us consider the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ from the North pole.

(a) Let $L \subseteq S^2$ be a line. Show that the image $\pi_N(L)$ is a line if and only if L passes through the North pole.

(b) Let $L \subseteq S^2$ be a line such that $N \notin L$. Show that $\pi_N(L)$ is a circle.

(c) Show that π_N is *not* an isometry.

(d) Prove that π_N preserves angles.

Problem 5. (20 pts) **Stereographic Projection for 1-sphere**. Let us consider the 1-sphere $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with its North Pole $N = (0, 1)$, and $H = \{(x, y) \in \mathbb{R}^2 : y = 0\} \subseteq \mathbb{R}^2$ the horizontal x-axis. Define the stereographic projection

$$\pi_N : S^1 \setminus \{N\} \longrightarrow \mathbb{R}, \quad \pi_N(P) = Q,$$

where $P = (x, y) \in S^1$, and $Q = u \in \mathbb{R}$ is defined as the unique intersection point $L_{P,N} \cap H$ distinct from N, and $L_{P,N} \subseteq \mathbb{R}^2$ is the unique line containing $P, N \in \mathbb{R}^2$.

(a) Find a formula of $\pi_N(P) = u$ in terms of $P = (x, y)$.

(b) Show that $(x, y) \in S^1 \setminus \{N\}$ has rational coordinates, i.e. $x, y \in \mathbb{Q}$, if and only if $\pi_N(x, y)$ is a rational number.

(c) Show that $\pi_N : (S^1 \setminus \{N\}) \cap \mathbb{Q}^2 \longrightarrow \mathbb{Q}$ is a bijection.

(d) Show that the inverse image $\pi_N^{-1}(u)$ of a rational point $u = m/n \in \mathbb{Q} \subseteq \mathbb{R}$, $m, n \in \mathbb{N}$, is given by

$$(x, y) = \left(\frac{2nm}{n^2 + m^2}, \frac{m^2 - n^2}{n^2 + m^2} \right).$$

(e) Let $m, n \in \mathbb{N}$ be two natural numbers, and define

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

Show that $a^2 + b^2 = c^2$.

(f) *Euclid's Theorem For Phytagorean Triples.* (Optional: Not graded)
Let $a, b, c \in \mathbb{N}$ be a primitive Phytagorean triple, i.e. $a^2 + b^2 = c^2$ such that $\gcd(a, b, c) = 1$. Show that there exist $m, n \in \mathbb{N}$ such that

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$