University of California Davis
Euclidean Geometry MAT 141

Name (Print): ______________________
Student ID (Print): ______________________

Sample Final Examination
Time Limit: 120 Minutes

March 17 2020

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Euclidean Geometry**) Consider the three Euclidean lines in $\mathbb{R}^2$ given by $L = \{(x, y) \in \mathbb{R}^2 : x = 0\}$, $M = \{(x, y) \in \mathbb{R}^2 : y = 3\}$ and $N = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$.

   (a) (5 points) Find the image of $(0, 3) \in \mathbb{R}^2$ under the isometry $\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L$.

   (b) (5 points) Let $R = \{(x, y) \in \mathbb{R}^2 : y = x + 3\} \subseteq \mathbb{R}^2$ be a line and $P \in R$ an arbitrary point in the line. Show that $(\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L)(P) \in R$.

   (c) (10 points) Find $\alpha \in \mathbb{R}^2$ and $K \subseteq \mathbb{R}^2$ a line such that

$$\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L = t_\alpha \circ \overline{r}_K.$$

2. (20 points) (**Spherical Geometry I**) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and its points $P = \frac{1}{\sqrt{2}}(1, 1, 0), Q = \frac{1}{\sqrt{2}}(1, 0, 1)$ and $R = \frac{1}{\sqrt{2}}(0, 1, 1)$.

(a) (5 points) Compute the three distances $d_{S^2}(P, Q)$, $d_{S^2}(R, Q)$ and $d_{S^2}(P, R)$.

(b) (10 points) Let $E \subseteq S^2$ be the unique line containing $P, Q \in S^2$. Show that

$$\Pi_E = \{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\}$$

is the unique 2-plane $\Pi_E \subseteq \mathbb{R}^3$ such that $\Pi_E \cap S^2 = E$.

(c) (5 points) Find the image $\overline{r}_E(R)$ of R under the reflection $\overline{r}_E : S^2 \longrightarrow S^2$.

3. (20 points) (**Spherical Geometry II**) Consider the unit 2-sphere S^2 and the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ from the North Pole $N = (0, 0, 1)$. Consider the two points $P_1 = (1, 0, 0), P_2 = \frac{1}{\sqrt{3}}(1, -1, 1)$ and let $L \subseteq S^2$ be the unique line which contains P_1, P_2.

   (a) (5 points) Find the image $\pi_N(P_1)$ and $\pi_N(P_2)$ of the two points

$$P_1 = (1, 0, 0), P_2 = \frac{1}{\sqrt{3}}(1, -1, 1).$$

   (b) (5 points) Find the two intersection points $L \cap \mathcal{E}$, where $\mathcal{E}$ is the equator defined as $\mathcal{E} = \{(x, y, z) \in S^2 : z = 0\} \subseteq S^2$.

   (c) (5 points) Find all the points of intersection of $\pi_N(L) \subseteq \mathbb{R}^2$ and the unit circle $\{(x, y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$.

   (d) (5 points) Qualitatively draw the image $\pi_N(L) \subseteq \mathbb{R}^2$ in the Euclidean plane.

4. (20 points) (**Hyperbolic Geometry in** $\mathbb{H}^2$) Let $k \in \mathbb{N}$ and consider the curves $\gamma_k \subseteq \mathbb{H}^2$ described as

$$\gamma_k = \{(x, y) \in \mathbb{H}^2 : 0 \leq x \leq 1, y = k\} \subseteq \mathbb{H}^2.$$

(a) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$ between $P = (0, 1)$ and the point $Q = (1, 1)$.

(b) (5 points) Compute the length $l(\gamma_1)$ of γ_1, and use Part(a) to conclude that γ_1 is not part of a hyperbolic line.

Hint: Given a hyperbolic line L containing $P, Q \in \mathbb{H}^2$, the length of the bounded part in L between P and Q, must be the distance $d_{\mathbb{H}^2}(P, Q)$.

(c) (10 points) Show that the length $l(\gamma_k)$ strictly decreases as $k \to \infty$ increases.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) There exist no parallel hyperbolic lines in $(\mathbb{H}^2, d_{\mathbb{H}^2})$.

(1) True. (2) False.

(b) (2 points) Any two lines $L_1, L_2 \subseteq K$ in the Klein bottle must intersect once.

(1) True. (2) False.

(c) (2 points) A triangle $T \subseteq (\mathbb{H}^2, d_{\mathbb{H}^2})$ can have arbitrarily small interior angles.

(1) True. (2) False.

(d) (2 points) A triangle $T \subseteq (S^2, d_{S^2})$ can have three interior right angles.

(1) True. (2) False.

(e) (2 points) There exists a triangle $T \subseteq C$ in the Euclidean cylinder with two interior right angles.

(1) True. (2) False.

(f) (2 points) An isometry $f \in \text{Iso}(S^2)$ which fixes the three points $(0,0,1), \frac{1}{\sqrt{2}}(0,1,1), (0,1,0)$ in the 2-sphere S^2, must be the identity.

(1) True. (2) False.

(g) (2 points) Let $P, Q \in S^2$ be two points in the 2-sphere S^2, then there exists a unique line $L \subseteq S^2$ containing P and Q.

(1) True. (2) False.

(h) (2 points) The product of two reflections $\overline{r}_1, \overline{r}_2 \in \text{Iso}(S^2)$ is a rotation.

(1) True. (2) False.

(i) (2 points) The product of two reflections $\overline{r}_1, \overline{r}_2 \in \text{Iso}(\mathbb{R}^2)$ is a rotation.

(1) True. (2) False.

(j) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ sends triangles in S^2 to triangles in $\mathbb{R}^2$.

(1) True. (2) False.