University of California Davis **Name (Print):** ____________________

Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Sample Final Examination II **March 17 2020**

Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Isometries of Euclidean Plane** $\mathbb{R}^2$). Let $L = \{(x, y) \in \mathbb{R}^2 : x = y\} \subseteq \mathbb{R}^2$ be a line, and $f = t_{(1,1)} \circ \overline{r}_L$ a glide reflection along L.

   (a) (5 points) Show that $f(P) \neq P$ for any $P \in \mathbb{R}^2$, i.e. f has no fixed points.

   (b) (10 points) Let $M = \{(x, y) \in \mathbb{R}^2 : y = 0\} \subseteq \mathbb{R}^2$ be the horizontal line. Show that the isometry $f \circ \overline{r}_M$ is a rotation and find its center.

   (c) (5 points) Is it true that $f \circ \overline{r}_M = \overline{r}_M \circ f$?

2. (20 points) (**Isometries and Triangles in the 2-sphere S^2**) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and the three lines $L_1 = \{(x, y, z) \in \mathbb{R}^3 : y = 0\} \cap S^2$, $L_2 = \{(x, y, z) \in \mathbb{R}^3 : x = y\} \cap S^2$ and $L_3 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \cap S^2$.

(a) (5 points) Let $T \subseteq S^2$ be the triangle given by L_1, L_2, L_3 with vertices

$$(1, 0, 0), \frac{1}{\sqrt{2}}(1, 1, 0), (0, 0, 1).$$

Show that the angle α of T at $N = (0, 0, 1)$ is $\pi/4$.

(b) (5 points) Compute the area of the triangle T.

(c) (10 points) Find the area of the image $\pi_N(T)$ of $T \subseteq S^2$ under the stereographic projection $\pi_N : S^2 \setminus \{(0, 0, 1)\} \longrightarrow \mathbb{R}^2$.

3. (20 points) (**Distances and Lines in the 2-sphere** S^2) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3.$$

(a) (5 points) Compute the spherical distance between

$$P = (1, 0, 0) \in S^2 \text{ and } Q = \frac{1}{\sqrt{3}}(1, 1, 1) \in S^2.$$

(b) (10 points) Find an implicit equation for the line $L_{P,Q} \subseteq S^2$ defined as the set of equidistant points to $P, Q \in S^2$, i.e. $L_{P,Q} = \{R \in S^2 : d_{S^2}(R, P) = d_{S^2}(R, Q)\}$.

(c) (5 points) Find an isometry $\varphi : S^2 \longrightarrow S^2$ such that $\varphi(P) = Q$.

4. (20 points) (**Distances in the Hyperbolic Upper-Half Plane** $\mathbb{H}^2$) Let $P, Q \in \mathbb{H}^2$ be the points $P = (0, 2) = 2i$ and $Q = (2, 2) = 2 + 2i$.

   (a) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$.

   (b) (10 points) Find an implicit equation for the unique line $L \subseteq \mathbb{H}^2$ which contains $P, Q \in \mathbb{H}^2$ and draw a picture for it.

   (c) (5 points) Find an isometry $\phi : (\mathbb{H}^2, d_{\mathbb{H}^2}) \longrightarrow (\mathbb{H}^2, d_{\mathbb{H}^2})$ such that $\phi(P) = Q$.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

   (a) (2 points) Given a line $L \subseteq \mathbb{H}^2$ and a point $P \in \mathbb{H}^2$ not in the line, there exists a unique parallel line $M \subseteq \mathbb{H}^2$ to L containing P.

   (1) True. (2) False.

   (b) (2 points) Any Euclidean line $M \subseteq \mathbb{H}^2$ is a hyperbolic line.

   (1) True. (2) False.

   (c) (2 points) There exists a unique isometry $\varphi : S^2 \longrightarrow S^2$ which fixes both the North and the South pole.

   (1) True. (2) False.

   (d) (2 points) If two triangles $T_1, T_2 \subseteq \mathbb{R}^2$ have the same interior angles, then Area(T_1)=Area(T_2).

   (1) True. (2) False.

   (e) (2 points) If two triangles $T_1, T_2 \subseteq S^2$ have the same interior angles, then Area(T_1)=Area(T_2).

   (1) True. (2) False.

   (f) (2 points) For each $n \in \mathbb{N}$, there exists two lines $L_1, L_2 \subseteq T^2$ such that $|L_1 \cap L_2| = n$.

   (1) True. (2) False.

   (g) (2 points) There exists lines $L_1, L_2 \subseteq C$ in the cylinder C such that $|L_1 \cap L_2| = 5$.

   (1) True. (2) False.

   (h) (2 points) The Klein bottle K is locally isometric to the Möbius band.

   (1) True. (2) False.

   (i) (2 points) For any $P \in S^2$, there exists a disk $D \subseteq S^2$ such that D is isometric to an Euclidean disk in $\mathbb{R}^2$.

   (1) True. (2) False.

   (j) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ preserves distances.

   (1) True. (2) False.